Unit Name: Unit 2: Quadratic Functions and Modeling

Lesson Plan Number & Title: Lesson 2: The Changing Rate of Change

Grade Level: High School Math II

Lesson Overview:
Students will analyze functions using different representations. By calculating and interpreting average rate of change over several intervals, students will construct and compare linear, exponential, and polynomial models in terms of context. Students will observe that a quantity increasing exponentially eventually exceeds a quantity increasing as a linear or polynomial function.

Focus/Driving Question:
How can the rate of change of functions be useful in interpreting physical world situations?

West Virginia College- and Career-Readiness Standards:
M.2HS.9
Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. Instructional Note: Focus on quadratic functions; compare with linear and exponential functions studied in Mathematics I.

M.2HS.16
Using graphs and tables, observe that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically; or (more generally) as a polynomial function. Instructional Note: Compare linear and exponential growth studied in Mathematics I to quadratic growth.

Manage the Lesson:
Divide students into teams of 3 or 4 students.

Prepare a Resource/Learning Center for differentiating and tiering. Include the following possible tips or hints:
Average Rate of Change Help
Identifying Linear, Polynomial and Exponential Functions Help

As a homework assignment at the end of each day, each student will use a word processor to keep a daily writing journal that includes accomplishments and a reflection of lessons learned. All entries will be in complete sentences.

Students will complete the activities in the Launch/Introduction and Investigate/Explore sections in 90 minute periods as follows:
Step 1 -- Launch/Introduction activity; Vocabulary and Concept Development Activity
Step 2 -- Review Linear Data Activities
Step 3 -- Investigating Average Rate of Change Activity
Step 4 -- Rate of Change Applications Activity

After students have completed an activity, the teacher will lead a discussion either with individual groups or as whole class discourse.

Differentiation: Classroom format includes a mix of whole group, collaborative group, paired and individual activities. Functions are modeled in a wide variety of ways using physical and virtual manipulatives, graphing technology and Internet web sites. All explorations and discovery activities offer a variety of entry points. A Resource/Learning Center is provided that includes materials to meet the needs of all learners. Step-by-step instructions should be provided for the special needs student.

Academic Vocabulary Development:

average rate of change
decreasing function
exponential function
increasing function
line of best fit
linear function
polynomial function
quadratic function
scatterplots
secant line

Most of the vocabulary was developed and used extensively in the Math I course. The remaining terms will be addressed in the Lesson 2 - Vocabulary and Concept Development activity. The teacher will reinforce the vocabulary development from Math I and the activity by using the terms during the whole class discourse after each of the discovery and investigation activities. The students will demonstrate mastery of vocabulary by using each of the terms properly in their activities and daily journal entries.

Launch/Introduction:

At the beginning of the lesson, the teacher will project Oil Change vs. Engine Repairs from Illuminations: Exploring Linear Data - http://illuminations.nctm.org/LessonDetail.aspx?id=L298 and follow the instructional plan from Illuminations reviewing vocabulary, constructing scatterplots, interpreting data points and trends, and investigating line of best fit. In their groups, students will discuss and individually complete the activity sheets. The teacher will assist individual groups as needed with the activity.

Investigate/Explore:

Vocabulary and Concept Development Activity
Students will research the internet and complete the Lesson 2 - Vocabulary and Concept Development activity.
Suggested definitions and concepts can be found at Lesson 2 - Vocabulary and Concept Development Key.
Remind the students that they need to use each of the terms and concepts properly in their activities and daily journal entries.

Review Linear Data Activities
The teacher will distribute the Bike Weights and Jump Heights activity sheet, and Weights and Drug Doses activity sheet from Illuminations: Exploring Linear Data - http://illuminations.nctm.org/LessonDetail.aspx?id=L298 and follow the instructional plan from Illuminations reviewing vocabulary, constructing scatterplots, interpreting data points and trends, and investigating line of best fit. In their groups, students will discuss and individually complete the activity sheets.

Investigating Average Rate of Change Activity
Students with the aid of a graphing calculator and/or CAS (Computer Algebra System) will investigate average rate of change of different types of functions. In their groups, students will discuss and individually complete Investigating Average Rate of Change activity.
Suggested solutions can be found at Investigating Average Rate of Change Key.

Rate of Change Applications Activity
Students will apply average rate of change to different types of functions in various forms. In their groups, students will discuss and individually complete Rate of Change Applications activity.
Suggested solutions can be found at Rate of Change Applications Key.

Summarize/Debrief:
After students have completed each activity, the teacher will lead a discussion either with individual groups or as whole class discourse. The teacher will use students' responses from activities and their responses during discussion to determine concepts that need to be retaught and revisited.

Materials:
Graph paper
Graphing calculator and/or CAS (Computer Algebra System)
Illuminations: Exploring Linear Data - http://illuminations.nctm.org/LessonDetail.aspx?id=L298
XP - Math Jobs - http://www.xpmath.com/careers/math_jobs.php
We Use Math - http://weusemath.org/
Average Rate of Change Help
Identifying Linear, Polynomial and Exponential Functions Help
Lesson 2 - Vocabulary and Concept Development
Lesson 2 - Vocabulary and Concept Development Key
Investigating Average Rate of Change
Investigating Average Rate of Change Key
Rate of Change Applications
Rate of Change Applications Key

Career Connection:
Careers in the Science, Technology, Engineering and Mathematics cluster that use rates of change and exponential growth are actuaries, data analysts, mathematicians and statisticians along with several research careers in the Health Science cluster including biotechnology research and development. For more information on mathematics jobs and careers, see XP - Math Jobs - http://www.xpmath.com/careers/math_jobs.php and We Use Math - http://weusemath.org/.

Lesson Reflection:
The teacher will reflect on how the lesson went and determine the parts from the entire lesson that need to be revised or revisited.
As a final entry in their daily journal, students will respond to the following:
As you reflect on this lesson, in what ways were you most successful? In what areas do you still need improvement? Justify your responses.

Average Rate of Change Help

Comparing Properties of Linear and Exponential Functions (Math I) Utah Electronic High School -- https://share.ehs.uen.org/node/21329

Increasing and Decreasing Functions -- http://www.mathsisfun.com/sets/functions-increasing.html

Identifying Linear, Polynomial and Exponential Functions Help

Comparing Linear, Quadratic, and Exponential Functions Tutorial -- http://www.sophia.org/comparing-linear-quadratic-and-exponential-functions/comparing-linear-quadratic-and-exponential-functio--3-tutorial?pathway=nc-skill-366

Nonlinear Functions -- http://www.montereyinstitute.org/courses/Algebra1/COURSE_TEXT_RESOURCE/U03_L2_T5_text_final.html

Vocabulary and Concept Development

Slope is often referred to as rate of change. Why is the rate of change for any given line always constant?

What is the definition of a secant line?

Suppose P and Q are distinct points on the graph of a function $f(x)$ such that $P=(a, f(a))$ and $Q=(x, f(x))$. Find the slope of the secant line $\overleftrightarrow{PQ}$.

Define average rate of change.

Define increasing function over (a,b).

Define decreasing function over (a,b).

Vocabulary and Concept Development Key

Slope is often referred to as rate of change. Why is the rate of change for any given line always constant?

The slope (rate of change) of a line is $m = \frac{\Delta y}{\Delta x}$. The quotient $\frac{y_2 - y_1}{x_2 - x_1}$ is the same ratio for any distinct points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ on the graph of the line.

What is the definition of a secant line?
A secant line is a line which passes through at least two points of a curve.

Suppose P and Q are distinct points on the graph of a function $f(x)$ such that $P = (a, f(a))$ and $Q = (x, f(x))$. Find the slope of the secant line $\overleftrightarrow{PQ}$.

$$m_{\overleftrightarrow{PQ}} = \frac{f(x) - f(a)}{x - a}$$

Define average rate of change.
Average rate of change of a function is change in the y-values divided by the change in the x- values for two distinct points on the graph of the function. That is, average rate of change is the slope of the secant line that passes through the two points.

Define increasing function over (a,b).

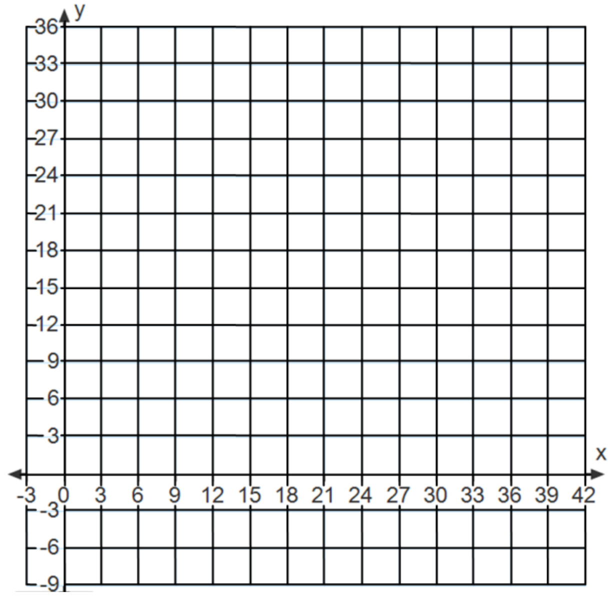
A function f is increasing over (a,b) if $x_1 < x_2$ then $f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in (a,b)$.

Define decreasing function over (a,b).

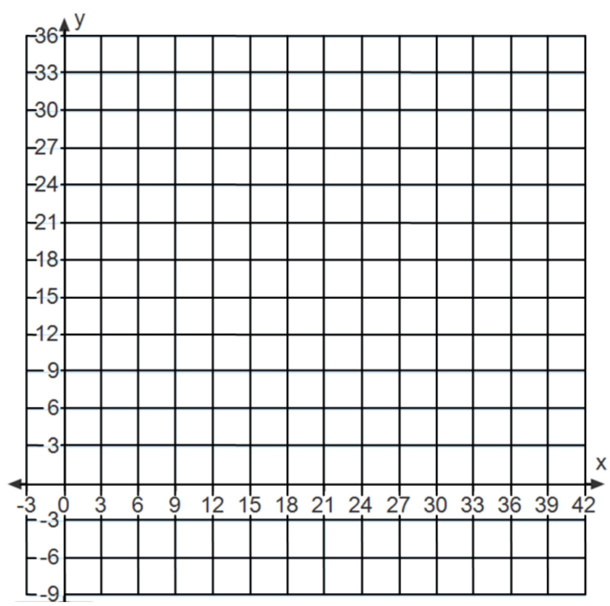
A function f is decreasing over (a,b) if $x_1 < x_2$ then $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in (a,b)$.

Investigating Average Rate of Change

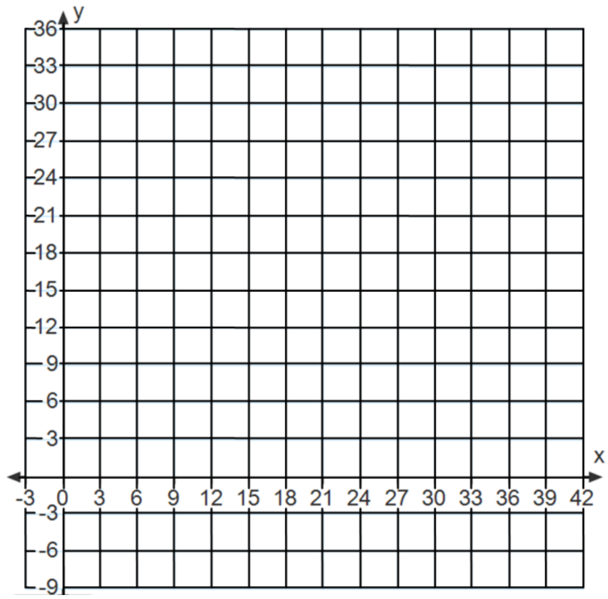
Use a table of values to sketch the graph of the function $k(x)=3x-9$ and check using a graphing calculator and/or CAS (Computer Algebra System). Find the average rate of change over each of the following intervals: $[3,6]$, $[6,9]$, $[9,12]$.



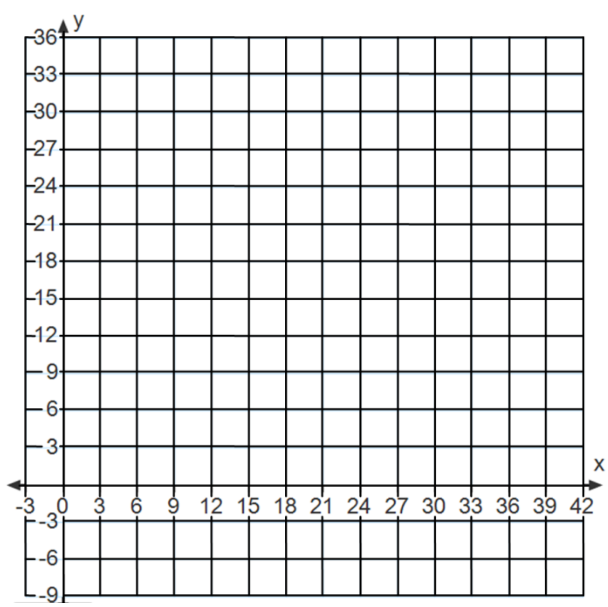
Use a table of values to sketch the graph of the function $q(x)=x^2-13x+36$ and check using a graphing calculator and/or CAS. Find the average rate of change and sketch the graph of the secant lines over each of the following intervals: $[3,6]$, $[6,9]$, $[9,12]$.



Use a table of values to sketch the graph of the function $p(x) = x^3 - 18x^2 + 104x - 192$ and check using a graphing calculator and/or CAS. Find the average rate of change and sketch the graph of the secant lines over each of the following intervals: $[3, 6]$, $[6, 9]$, $[9, 12]$.



Use a table of values to sketch the graph of the function $r(x) = (1.25)^x$ and check using a graphing calculator and/or CAS. Find the average rate of change and sketch the graph of the secant lines over each of the following intervals: $[3, 6]$, $[6, 9]$, $[9, 12]$. (Estimate values to the nearest hundredth.)



Use information from the preceding problems to compare the average rate of change for the functions $k(x)=3x-9$, $q(x)=x^2-13x+36$, $p(x)=x^3-18x^2+104x-192$ and $r(x)=(1.25)^x$ over the intervals $[3,6]$, $[6,9]$, $[9,12]$ and $[10^{100},10^{101}]$. For each of the intervals in the following table, rank the functions in order from greatest average rate of change to lowest.

$[3,6]$	$[6,9]$	$[9,12]$	$[10^{100},10^{101}]$

Consider linear, quadratic, polynomial and exponential functions that are increasing as x approaches infinity. In general, what type of function increases faster as x approaches infinity? Explain your reasoning.

Consider linear, quadratic, polynomial and exponential functions that are increasing as x approaches infinity. In general, what type of function exceeds the values of the other types of functions as x approaches infinity? Explain your reasoning.

Investigating Average Rate of Change Key

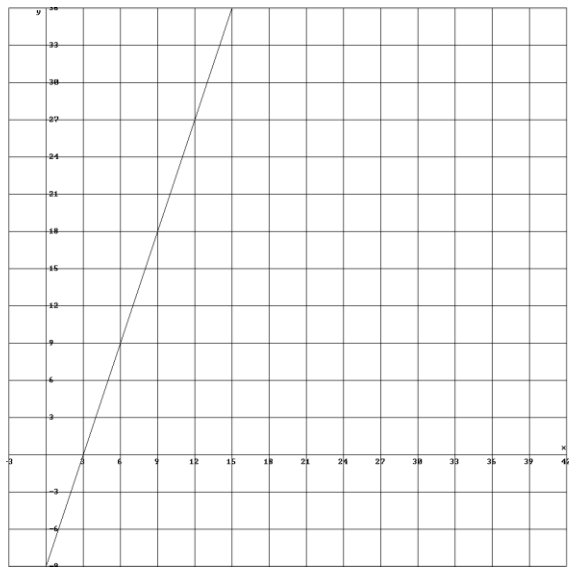
Use a table of values to sketch the graph of the function $k(x) = 3x - 9$ and check using a graphing calculator and/or CAS (Computer Algebra System). Find the average rate of change over each of the following intervals: $[3,6]$, $[6,9]$, $[9,12]$.

x	$k(x)$
0	-9
3	0
6	9
9	18
12	27

$$m_{[3,6]} = \frac{9-0}{6-3}$$
$$m_{[3,6]} = 3$$

$$m_{[6,9]} = \frac{18-9}{9-6}$$
$$m_{[6,9]} = 3$$

$$m_{[9,12]} = \frac{27-18}{12-9}$$
$$m_{[9,12]} = 3$$



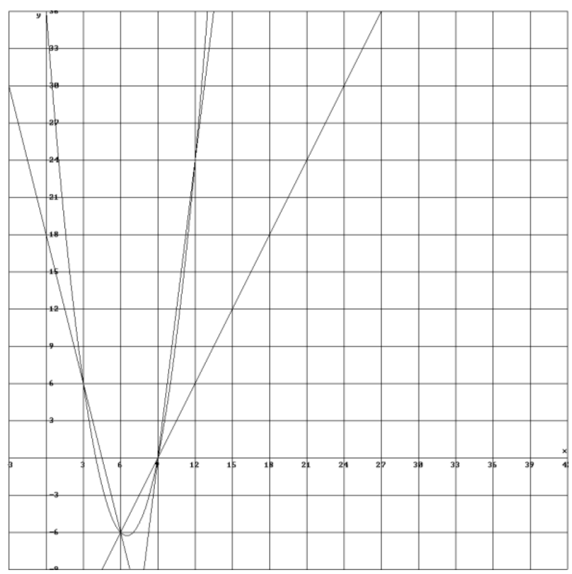
Use a table of values to sketch the graph of the function $q(x) = x^2 - 13x + 36$ and check using a graphing calculator and/or CAS. Find the average rate of change and sketch the graph of the secant lines over each of the following intervals: $[3,6]$, $[6,9]$, $[9,12]$.

x	$q(x)$
0	36
3	6
4	0
6	-6
$\frac{13}{2}$	$-\frac{25}{4}$
7	-6
9	0
12	24

$$m_{[3,6]} = \frac{-6-6}{6-3}$$
$$m_{[3,6]} = -4$$

$$m_{[6,9]} = \frac{0+6}{9-6}$$
$$m_{[6,9]} = 2$$

$$m_{[9,12]} = \frac{24-0}{12-9}$$
$$m_{[9,12]} = 8$$



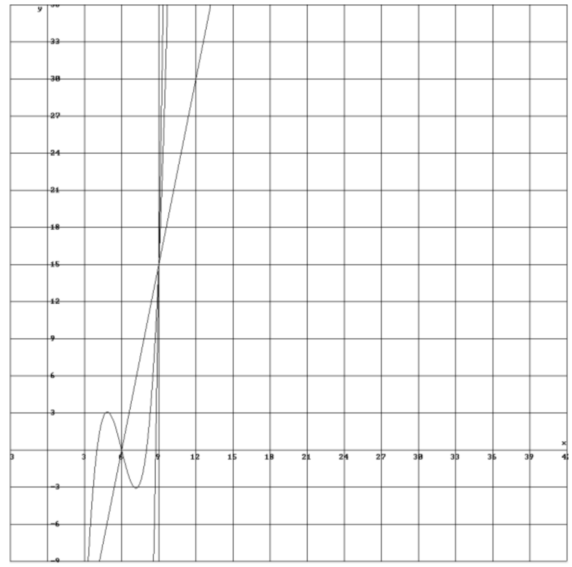
Use a table of values to sketch the graph of the function $p(x) = x^3 - 18x^2 + 104x - 192$ and check using a graphing calculator and/or CAS. Find the average rate of change and sketch the graph of the secant lines over each of the following intervals: $[3,6]$, $[6,9]$, $[9,12]$.

x	$p(x)$
0	-192
3	-15
4	0
5	3
6	0
7	-3
8	0
9	15
12	192

$$m_{[3,6]} = \frac{0+15}{6-3}$$
$$m_{[3,6]} = 5$$

$$m_{[6,9]} = \frac{15-0}{9-6}$$
$$m_{[6,9]} = 5$$

$$m_{[9,12]} = \frac{192-15}{12-9}$$
$$m_{[9,12]} = 59$$



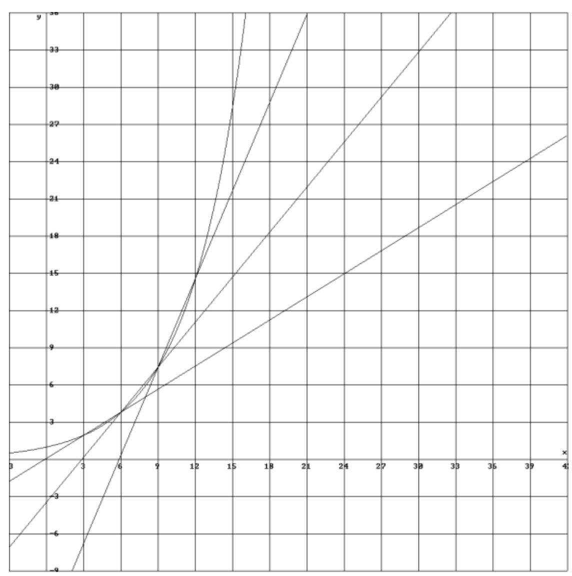
Use a table of values to sketch the graph of the function $r(x) = (1.25)^x$ and check using a graphing calculator and/or CAS. Find the average rate of change and sketch the graph of the secant lines over each of the following intervals: $[3,6]$, $[6,9]$, $[9,12]$. (Estimate values to the nearest hundredth.)

x	$r(x)$
0	1
3	1.95
6	3.81
9	7.45
12	14.55
15	28.42

$$m_{[3,6]} \approx \frac{3.81-1.95}{6-3}$$
$$m_{[3,6]} \approx 0.62$$

$$m_{[6,9]} \approx \frac{7.45-3.81}{9-6}$$
$$m_{[6,9]} \approx 1.21$$

$$m_{[9,12]} \approx \frac{14.55-7.45}{12-9}$$
$$m_{[9,12]} \approx 2.37$$



Use information from the preceding problems to compare the average rate of change for the functions $k(x)=3x-9$, $q(x)=x^2-13x+36$, $p(x)=x^3-18x^2+104x-192$ and $r(x)=(1.25)^x$ over the intervals $[3,6]$, $[6,9]$, $[9,12]$ and $[10^{100},10^{101}]$. For each of the intervals in the following table, rank the functions in order from greatest average rate of change to lowest.

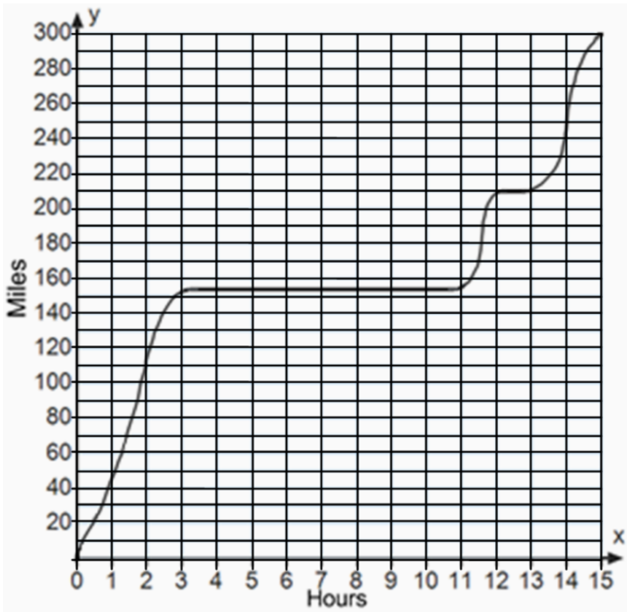
$[3,6]$	$[6,9]$	$[9,12]$	$[10^{100},10^{101}]$
$p(x)=x^3-18x^2+104x-192$	$p(x)=x^3-18x^2+104x-192$	$p(x)=x^3-18x^2+104x-192$	$r(x)=(1.25)^x$
$k(x)=3x-9$	$k(x)=3x-9$	$q(x)=x^2-13x+36$	$p(x)=x^3-18x^2+104x-192$
$r(x)=(1.25)^x$	$q(x)=x^2-13x+36$	$k(x)=3x-9$	$q(x)=x^2-13x+36$
$q(x)=x^2-13x+36$	$r(x)=(1.25)^x$	$r(x)=(1.25)^x$	$k(x)=3x-9$

Consider linear, quadratic, polynomial and exponential functions that are increasing as x approaches infinity. In general, what type of function increases faster as x approaches infinity? Explain your reasoning.
As x becomes large, an increasing exponential function increases faster than linear, quadratic and polynomial functions.
For consecutive integer values of x, the value of a linear function increases by a constant. In other words, for large consecutive integer values of x, the values of a linear function increases by a factor closer and closer to one. Likewise, for large consecutive integer values of x, the values of both quadratic and polynomial functions increases by a factor closer and closer to one. Whereas for increasing function $f(x)=Ab^x, b>0$ and the values of $f(x)$ increases by a factor of b for all consecutive integer values of x.

Consider linear, quadratic, polynomial and exponential functions that are increasing as x approaches infinity. In general, what type of function exceeds the values of the other types of functions as x approaches infinity? Explain your reasoning.
As x becomes large, an increasing exponential function exceeds the values of the other types of functions as x approaches infinity.
As x becomes large, the average rate of change of an exponential function increases faster than linear, quadratic and polynomial functions. Furthermore, all of these functions are continuous. In conclusion, increasing exponential functions exceed the values of other types of functions as x approaches infinity.

Rate of Change Applications

1. On Wednesday, around 8:00 PM, Mr. and Mrs. Saib reported that their daughter, Sara, is missing. As of 7:00 PM, Wednesday, she had not checked into her beach house rental. They reported that she left their house at 7:00 PM, Tuesday, to go on vacation. The police found her abandoned vehicle with her GPS. A sketch of the graph from her GPS is below. Use information from the sketch to answer the following questions. (Approximate to the nearest tenth, as needed.)



a. What was the total distance traveled?

b. How far did Sara travel by 10:00 PM, Tuesday?

c. How far did she travel during the last hour of her trip?

d. What is the most common term that people use for average rate of change in this type of problem?

e. What was her average rate of change from 7:00 PM to 10:00 PM?

f. What was her average rate of change for the entire trip?

g. What was her average rate of change during the last four hours of the trip?

h. Using your knowledge of average rate of change and information from Sara's GPS, what do you find unusual about her trip?

i. Use the information from the graph and your knowledge of average rate of change to write a paragraph that describes Sara's trip in detail.

2. Use the following table of Mathematics SAT Scores by Gender to answer the following questions.

Year	Male	Female
1975	518	479
1980	515	473
1985	522	480
1990	521	483
1995	525	490
2000	533	498
2005	538	504
2010	533	499

a. Find the average rate of change for the male scores from 2000 to 2010 and interpret its meaning.

b. Find the average rate of change for the female scores from 2000 to 2010 and interpret its meaning.

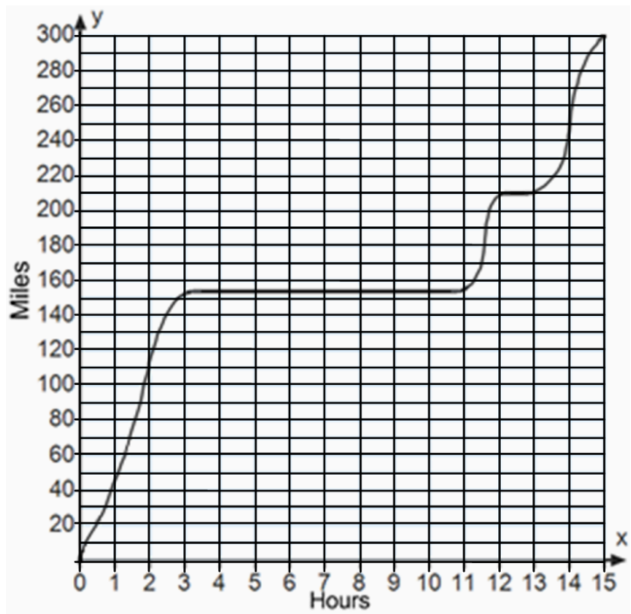
c. Complete the fourth column in the following table that represents the difference in the male and female scores. Find the average rate of change for the difference in scores from 1975 to 2010 and interpret its meaning. (Approximate to the nearest hundredth.)

Year	Male	Female	Difference (M - F)
1975	518	479	
1980	515	473	
1985	522	480	
1990	521	483	
1995	525	490	
2000	533	498	
2005	538	504	
2010	533	499	

3. According to rumors during a recent Mountaineer Balloon Festival in Morgantown, WV, world renowned balloonist, Betrand Piccard dropped his camera from his hot air balloon at an altitude of 900 feet. Find the average rate of change of the camera from 3 to 6 seconds and interpret its meaning. Hint: The equation for a free falling object that is dropped from 900 feet is $h(t) = -16t^2 + 900$.

Rate of Change Applications Key

1. On Wednesday, around 8:00 PM, Mr. and Mrs. Saib reported that their daughter, Sara, is missing. As of 7:00 PM, Wednesday, she had not checked into her beach house rental. They reported that she left their house at 7:00 PM, Tuesday, to go on vacation. The police found her abandoned vehicle with her GPS. A sketch of the graph from her GPS is below. Use information from the sketch to answer the following questions. (Approximate to the nearest tenth, as needed.)



a. What was the total distance traveled?
300 miles

b. How far did Sara travel by 10:00 PM, Tuesday?
155 miles

c. How far did she travel during the last hour of her trip?
50 miles

d. What is the most common term that people use for average rate of change in this type of problem?
Rate or average speed

e. What was her average rate of change from 7:00 PM to 10:00 PM?

$$m_{[0,3]} = \frac{155-0}{3-0}$$

$$m_{[0,3]} \approx 51.7mph$$

f. What was her average rate of change for the entire trip?

$$m_{[0,15]} = \frac{300-0}{15-0}$$

$$m_{[0,15]} = 20mph$$

g. What was her average rate of change during the last four hours of the trip?

$$m_{[11,15]} = \frac{300-155}{15-11}$$

$$m_{[11,15]} \approx 36.3mph$$

h. Using your knowledge of average rate of change and information from Sara's GPS, what do you find unusual about her trip?
Sara was travelling at excessive speed from approximately 6:30 AM to 6:45 AM, Wednesday. Her average speed was $m_{[11.5,11.75]} = \frac{200-170}{11.75-11.5}$ which is $120mph$.
Sara was travelling at excessive speed from approximately 8:45 AM to 9:15 AM, Wednesday. Her average speed was $m_{[13.5,14.25]} = \frac{270-230}{14.25-13.75}$ which is $80mph$.

i. Use the information from the graph and your knowledge of average rate of change to write a paragraph that describes Sara's trip in detail.
Answers will vary. Be aware that some students may misconceive the graph of distance as a function of time. They may incorrectly describe Sara's trip as though they are looking at an aerial view of a map of the roads that Sara traveled instead of the graph of a position function.

2. Use the following table of Mathematics SAT Scores by Gender to answer the following questions.

Year	Male	Female
1975	518	479
1980	515	473
1985	522	480
1990	521	483
1995	525	490
2000	533	498
2005	538	504
2010	533	499

a. Find the average rate of change for the male scores from 2000 to 2010 and interpret its meaning.

$$m_{M_{[2000-2010]}} = \frac{533-533}{2010-2000}$$

$$m_{M_{[2000-2010]}} = 0$$

Zero average rate of change means that the male mathematics SAT scores increased an average of zero each year from 2000 to 2010.

b. Find the average rate of change for the female scores from 2000 to 2010 and interpret its meaning.

$$m_{F_{[2000-2010]}} = \frac{499-498}{2010-2000}$$

$$m_{F_{[2000-2010]}} = .1$$

One tenth average rate of change means that the female mathematics SAT scores increased an average of 0.1 each year from 2000 to 2010.

c. Complete the fourth column in the following table that represents the difference in the male and female scores. Find the average rate of change for the difference in scores from 1975 to 2010 and interpret its meaning. (Approximate to the nearest hundredth.)

Year	Male	Female	Difference (M - F)
1975	518	479	39
1980	515	473	42
1985	522	480	42
1990	521	483	38
1995	525	490	35
2000	533	498	35
2005	538	504	34
2010	533	499	34

$$m_{M-F_{[1975-2010]}} = \frac{34-39}{2010-1975}$$

$$m_{M-F_{[1975-2010]}} \approx -0.14$$

Negative fourteen hundredths average rate of change means that the difference in the male and female mathematics SAT scores decreased an average of approximately 0.14 each year from 1975 to 2010.

3. According to rumors during a recent Mountaineer Balloon Festival in Morgantown, WV, world renowned balloonist, Betrand Piccard dropped his camera from his hot air balloon at an altitude of 900 feet. Find the average rate of change of the camera from 3 to 6 seconds and interpret its meaning. Hint: The equation for a free falling object that is dropped from 900 feet is $h(t) = -16t^2 + 900$.

$$m_{[3,6]} = \frac{h(6)-h(3)}{6-3}$$

$$m_{[3,6]} = \frac{-16\cdot 6^2 + \cancel{900} - \left(-16\cdot 3^2 + \cancel{900}\right)}{3}$$

$$m_{[3,6]} = \frac{-16(36-9)}{3}$$

$$m_{[3,6]} = \frac{-16\cdot 27}{3}$$

$$m_{[3,6]} = -16\cdot 9$$

$$m_{[3,6]} = -144$$

The negative indicates that the camera is falling. The camera is increasing speed as it is falling. That is, for each second the camera is falling from 3 to 6 seconds, it increases in speed an average of 144 feet per second.