Lesson Plan
Lesson 12: Medians and Proportionality
Mathematics High School Math II

Unit Name: Unit 5: Similarity, Right Triangle Trigonometry, and Proof

Lesson Plan Number & Title: Lesson 12: Medians and Proportionality

Grade Level: High School Math II

Lesson Overview: Students develop their ability to use coordinates to apply geometric theorems. Given a ratio, students use their understanding of geometric theorems to partition a line segment in the coordinate plane. This lesson is designed for approximately 90 to 120 minutes, but time may vary depending on the background of the students.

Focus/Driving Question: How can geometric theorems be applied algebraically? How can geometric theorems be applied to easily partition a given length into any number of equal parts?

West Virginia College- and Career-Readiness Standards:

M.2HS.47 Find the point on a directed line segment between two given points that partitions the segment in a given ratio

Manage the Lesson:

Through exploration and investigation, students apply their understanding of geometric theorems to divide a line segment into a given ratio.

Academic Vocabulary Development:

No new vocabulary is introduced. Students strengthen their understanding of related vocabulary in creating connections between algebra and geometry.

Launch/Introduction:

Introduce students to the following problem situation: Carpenters often need to divide a board or distance into segments of equal lengths. If, for instance, a carpenter needs to divide a ten-foot board into two segments of equal length, he would have little difficulty creating the two five-foot segments. Similarly, the carpenter would have little difficulty dividing the ten-foot board into five segments of equal length. Considering a ten-foot length as a length of 120 inches allows a carpenter to easily determine the length of segments when the number needed is a factor 120 (i.e., 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, or 120). Difficulty arises, however, when the number of segments needed is not a factor of 10 or 120.

Divide the students into teams and challenge them to devise a method to easily divide a ten-foot board into seven segments of equal length. If possible, post a ten-foot item on the wall. Provide each student team with an item (such as a length of ribbon) to divide into the seven segments. Determine a date by which student teams will need to present a written explanation of their method. Student teams will also need to generalize the method to easily allow a carpenter to divide a board of any length into any given number of segments. Provide student teams with access to a variety of tools (tape measures or rulers, protractors, compasses, etc.). Allow student teams an opportunity to develop an understanding of the problem situation.

Students may choose to search the Internet for possible solutions to this challenge. If students are struggling with this task, the website *TutorVista.com*, (http://math.tutorvista.com/geometry/proportionality-theorem.html) presents a step-by-step method of construction for dividing a line segment into equal parts. The website *Fine Woodworking for your Home, Geometry- a tool for the carpenter*, (http://www.fine-woodworking-for-your-home.com/Geometry.html) presents a method to divide a board or line into any number of equal parts.

Investigate/Explore:

Distribute 12.1 Medians. After student pairs have completed their investigations, in a whole class discussion, allow students to share and compare their findings. Some of the statements that students are asked to create reiterate previously proven theorems. In other statements, students pose conjectures about possible relationships. Prompt students to differentiate between conclusions and conjectures. Reinforce their ability to recognize the difference between conclusions and conjectures by asking students to identify each created statement as either a proven conclusion or a conjecture that needs to be proved. Ask students to explain their reasoning.

Challenge students to create a proof of one of the created conjectures: The midsegment of a triangle is parallel to the third side and half the length of the third side. 12.2 Midsegments of a Triangle provides two possible proofs. The first proof involves creating auxiliary lines. It may be helpful to suggest these lines to struggling students. The second proof involves rotating a triangle formed by the midsegments of the triangle to form a quadrilateral.

Challenge students to create a proof of the second created conjecture: The three midsegments of a triangle form four congruent triangles. Challenge students to also prove that these four congruent triangles are similar to the original triangle. In the **APPS MENU**, introduce students to Theorem 32 (Proportional Triangle Segments) -- If a segment divides two sides of a triangle proportionally, then it is in the same proportion to the third side and parallel to the third side. This theorem is a generalization of the first proof that students have been asked to create. Lead students to appreciate the simplicity of this transformational proof.

The corollary to Theorem 32 is the first proof that students were asked to create. Corollary 26.1 (Mid-segment Theorem) provides proofs of the student created conjectures. Examine these in a whole class format, encouraging students to supplement and explain the proofs.

Ask students to apply Theorem 26 (Proportional Triangle Segments) and Corollary 26.1(Mid-segment Theorem) with a construction. In a problem similar to the construction demonstrated in a step-by-step process at *TutorVista.com*, http://math.tutorvista.com/geometry/proportionality-theorem.html, provide students with a line segment and ask them to use a compass to divide it into five congruent segments. Provide students with additional construction tasks, dividing a given line segment into a given number of parts. 12.3 Dividing a Segment with GeoGebra asks students to apply their understanding of these theorems using dynamic geometric software. An example provides a step-by-step construction for dividing a given line segment into equal parts. (These construction tasks support the performance task introduced in the Lesson Launch.)

Ask students to prove the Converse of Theorem 32 - A line through two sides of a triangle and parallel to the third side divides the two sides proportionally. Students should quickly determine that the created triangles are similar and their corresponding sides are proportional. An interactive proof of this theorem can be found in the **APPS MENU** (Theorem 33 – Triangle Proportionality Theorem). The website http://www.kutasoftware.com/FreeWorksheets/GeoWorksheets/7-Proportional%20Parts%20in%20Triangles%20and%20Parallel%20Lines.pdf provides an opportunity for students to practice and apply their understanding of proportional triangles and parallel lines.

12.4 Triangle Medians prompts students to explore the relationship between the vertices of the triangle and the point of intersection of its three medians. (12.5 Investigation of Medians provides an second and more directed exploration of the relationship of the point of intersection of the medians of a triangle.) After students have developed an understanding of the relationship between the point of intersection of the medians and the vertices of the triangles, both the **APPS MENU** (Theorem 34) and 12.6 Triangle Medians Proof provide proofs verifying this relationship.

The website http://www.kutasoftware.com/FreeWorksheets/GeoWorksheets/5-Medians.pdf provides an opportunity for students to practice and apply their understanding of medians and the relationship between the medians in a triangle.

In a final investigation, students are asked to divide a line segment in the coordinate plane into a given number of segments and determine a method to identify the coordinates of the points of division. Given a line segment with endpoints (2, 6) and (8, 18), ask students to determine the midpoint of the segment. Ask students to justify their answers. In a whole class format, allow students to explain how they were able to demonstrate or prove that they have found the midpoint of the segment. Some students may choose to use the Pythagorean Theorem to determine the distance from each of the endpoints to the midpoint. Some students may create two right triangles, each with a hypotenuse extending from an endpoint to the midpoint and demonstrate that the hypotenuse of each of these triangles would be of equal length.

Continuing this investigation, challenge students to determine the coordinates of the point that is 1/3 the distance from (2, 6) to (8, 18). Encourage students to explain their reasoning and justify their solution. Now ask students to determine the coordinates of the point that is 1/4 the distance from (2, 6) to (8, 18). Again encourage students to explain and justify their solution.

Challenge students to create a rule for determining the coordinates of a point that is any distance d from (2, 6) to (8, 18). Lead students to create rules similar to $(2 + d(8 – 2), 8 + d(18 - 6))$. Students may express this verbally, saying that the x-coordinate of the point will be 2 + the fraction d of the horizontal distance between the two endpoints and the y-coordinate of the point will be 8 + the fraction of the vertical distance between the two endpoints). Finally ask students to generalize their rule to one that will give the coordinates of the point that is any distance r from (a, b) and (c, d). Lead students to create the coordinates $(a + r(c - a), b + r(d – b))$. A proof of this student-created rule is found in Theorem 35, Proportion of Directed Line Segment, in the **APPS MENU**.

Summarize/Debrief:

Student teams present their written explanations of the method they devised to easily allow a carpenter to divide a board of any length into any given number of segments. Combine two or three student pairs and in a round robin process, ask student teams to share and explain their process.

At this point, it may be beneficial to provide students with an interesting challenge. Ask students to devise a method to trisect an angle. Students may over generalize the procedure they developed for trisecting a segment. Help students appreciate that this construction technique does not result in three congruent angles. Students may find it interesting that it is not possible to trisect an arbitrary angle using only a compass and an unmarked straightedge and may wish to research the construction at sites such as http://mathworld.wolfram.com/AngleTrisection.html , http://terrytao.wordpress.com/2011/08/10/a-geometric-proof-of-the-impossibility-of-angle-trisection-by-straightedge-and-compass/, and http://www.math.tamu.edu/~mpilant/math646/MidtermProjects/Helgerud_midterm.pdf.

Materials:
Compasses
Patty paper
Tape measures or rulers
12.1 Medians
12.1 Medians – Key
12.2 Midsegments of a Triangle
12.3 Dividing a Segment with GeoGebra
12.4 Triangle Medians
12.4 Triangle Medians - Key
12.5 Investigation of Medians
12.6 Triangle Medians Proof

Proportional Parts in Triangles and Parallel Lines
An opportunity to demonstrate and practice skills
http://www.kutasoftware.com/FreeWorksheets/GeoWorksheets/7-Proportional%20Parts%20in%20Triangles%20and%20Parallel%20Lines.pdf
Medians in Triangles An opportunity to demonstrate and practice skills
http://www.kutasoftware.com/FreeWorksheets/GeoWorksheets/5-Medians.pdf
Fine Woodworking for your Home Geometry- a tool for the carpenter http://www.fine-woodworking-for-your-home.com/Geometry.html
Proportional Segments (proofs) http://my.safaribooksonline.com/book/geometry/9781598639841/ratios-and-proportions/ch03lev1sec2
Midline Theorem http://www.learner.org/courses/learningmath/geometry/session5/part_c/index.html
Four triangles created by midsegments are congruent
http://www.learner.org/courses/learningmath/geometry/session8/solutions_homework.html#h1
The Centroid http://math.kendallhunt.com/x3317.html
Trisect an Angle http://mathworld.wolfram.com/AngleTrisection.html
Trisect an Angle http://terrytao.wordpress.com/2011/08/10/a-geometric-proof-of-the-impossibility-of-angle-trisection-by-straightedge-and-compass/
Trisect an Angle http://www.math.tamu.edu/~mpilant/math646/MidtermProjects/Helgerud_midterm.pdf

Career Connection:

An ability to apply theoretical understanding in real-world problem situations is an essential skill in all career pathways. Those involved in construction find applying geometric understanding to real-world situations saves both time and effort.

Lesson Reflection:

Students should demonstrate an ability to use coordinates to apply geometric theorems. Given a line segment, students are able to partition the segment into a given number of equal parts.

In lesson 1, teachers were provided with a guide to aid them in reflecting upon the lesson as they seek to improve their practice. Certainly, it may not be feasible to formally complete such a reflection after every lesson, but hopefully the questions can generate some ideas for contemplation.

Medians

Triangle Mid-segments

1. Draw a relatively large scalene triangle on a piece of patty paper. Pinch the triangle to locate the midpoints of the sides.

2. Draw the mid-segments. This should create four small triangles.
(Recall that the segment that connects the midpoints of two sides of a triangle is a mid-segment. A triangle has three sides, each with its own midpoint, so there are three mid-segments in every triangle.)

3. Place a second piece of patty paper over the first and copy one of the triangles.

4. Compare all four triangles by sliding and/or rotating the copy of the one small triangle with the other three triangles.
Create a statement that describes the apparent relationship between the four =triangles.

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5. Identify and mark all the congruent angles in your drawing.
What conclusions does this allow you to make?
Create a statement relating the angles of the four small triangles and the angles of the large original triangle.

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Based on the relationship between the angles of the four small triangles and the angles of the original large triangle, create a statement relating the large original triangle and the four smaller triangles.

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6. Identify corresponding angles and alternate interior angles in your original drawing. What conclusions does this allow you to make?
Create a statement of your conclusions.
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7. Compare the length of the mid-segment to the large triangle's third side. What relationship appears to be true?
Create a statement of your conclusions.
____
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Medians - Key

Triangle Mid-segments

8. Draw a relatively large scalene triangle on a piece of patty paper. Pinch the triangle to locate the midpoints of the sides.

9. Draw the mid-segments. This should create four small triangles.
(Recall that the segment that connects the midpoints of two sides of a triangle is a mid-segment. A triangle has three sides, each with its own midpoint, so there are three mid-segments in every triangle.)

10. Place a second piece of patty paper over the first and copy one of the triangles.

11. Compare all four triangles by sliding and/or rotating the copy of the one small triangle with the other three triangles.
Create a statement that describes the apparent relationship between the four triangles.
The four triangles appear congruent.

12. Identify and mark all the congruent angles in your drawing.
What conclusions does this allow you to make?
Create a statement relating the angles of the four small triangles and the angles of the large original triangle.
The measures of the angles of the four small triangles are equal to the measures of the angles of the large original triangle.

    Based on the relationship between the angles of the four small triangles and the angles of the original large triangle, create a statement relating the large original triangle and the four smaller triangles.
The four small triangles are similar to the large original triangle.

13. Identify corresponding angles and alternate interior angles in your original drawing. What conclusions does this allow you to make?
Create a statement of your conclusions.
Each midsegment is parallel to the third side.

14. Compare the length of the mid-segment to the large triangle's third side.
What relationship appears to be true?
Create a statement of your conclusions.
The measure of the midsegment of a triangle is $\frac{1}{2}$ the measure of the third side.

Midsegments of a Triangle Proof

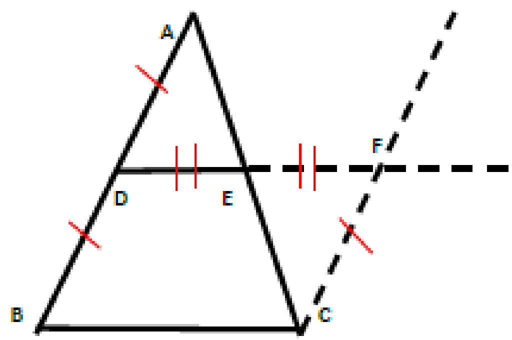
Midsegments of a Triangle (Proof 1)

Prove:

The midsegment of a triangle is parallel to the third side and half the length of the third side.

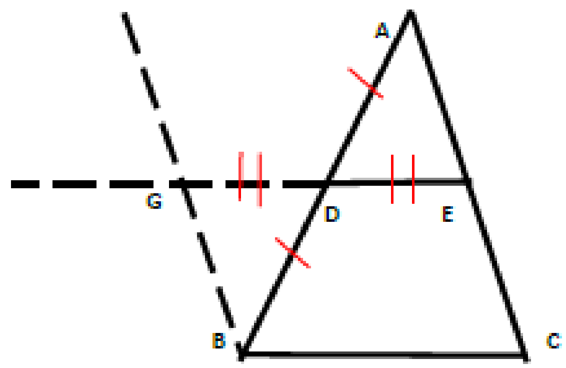
Given △ABC with D the midpoint of $\overline{AB}$ and E the midpoint of $\overline{AC}$, prove that $\overline{DE}$ is parallel to $\overline{BC}$ and DE is ½BC.

Create auxiliary line $\overleftrightarrow{CF} \parallel \overline{AB}$ and extend $\overleftrightarrow{DE}$ so that and $\overleftrightarrow{CF}$ and $\overleftrightarrow{DE}$ intersect at F.



- Given that $\overleftrightarrow{CF} \parallel \overline{AB}$, transversal DF creates congruent alternate interior angles ∠ADE and ∠EFC.
- Vertical angles ∠AED and ∠FEC are congruent.
- Because E is the midpoint of $\overline{AC}$, $\overline{AE} \cong \overline{EC}$.
- Therefore △AED ≅ △CEF.
- Because CPCTC, $\overline{DE} \cong \overline{EF}$ and $\overline{AD} \cong \overline{CF}$.
- Because D is the midpoint of $\overline{AB}$, $\overline{AD} \cong \overline{BD}$.
- Therefore, $\overline{BD} \cong \overline{CF}$.
- Because ∠ADE + ∠EDB = 180 and ∠EFC + ∠FCE + ∠CEF = 180,

  ∠ADE + ∠EDB = ∠EFC + ∠FCE + ∠CEF.
- By subtracting the equal ∠ADE and ∠EFC, results in ∠EDB = ∠FCE + ∠CEF
- Replacing ∠FCE + ∠CEF with ∠EDB, we have ∠EFC + ∠EDB = 180.



Similarly, creating auxiliary line $\overleftrightarrow{BG} \parallel \overline{AC}$ and extending $\overleftrightarrow{DE}$ so that and $\overleftrightarrow{CF}$ and $\overleftrightarrow{BG}$ intersect at G, yields we have ∠DBC + ∠FCB = 180.

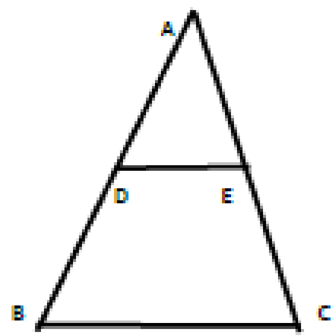
- Therefore, consecutive angles of DBCF are supplementary angles and DBCF is a parallelogram.
- Because DBCF is a parallelogram, GE = BC.
- Since GD + DE = GE = BC
- Since GD = DE, then DE + DE = BC and DE = ½BC

Midsegments of a Triangle (Proof 2)

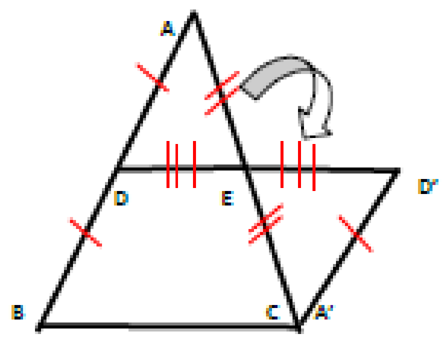
Prove:

The midsegment of a triangle is parallel to the third side and half the length of the third side.

Given △ABC with D the midpoint of $\overline{AB}$ and E the midpoint of $\overline{AC}$, prove that $\overline{DE}$ is parallel to $\overline{BC}$ and DE is ½BC.



Rotate △AD'C to create quadrilateral BDD'C.

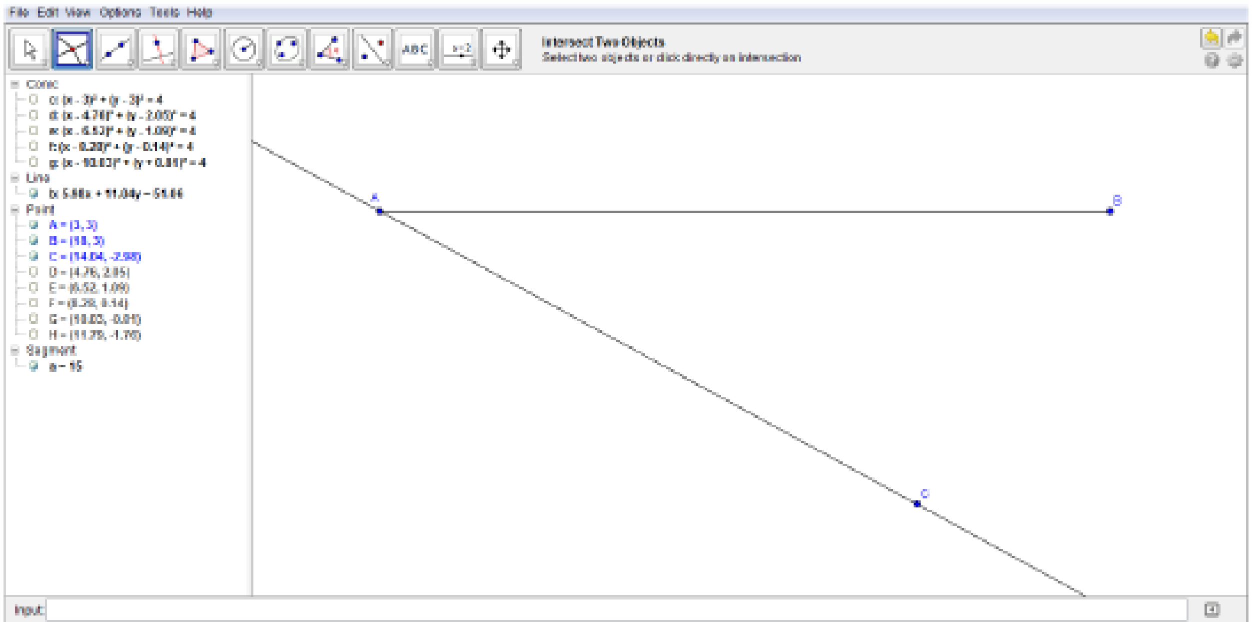


- $\overline{BD}$ is congruent to $\overline{D'C}$.
- ∠ADE and ∠ED'C are congruent.
- Because these alternate interior angles are congruent, $\overline{BD}$ || $\overline{D'C}$
- Since a pair of opposite sides of a BDD'A' are congruent and parallel, then BDD'A' is a parallelogram.
- Since opposites sides of a parallelogram are congruent, $\overline{BC}$ = 2$\overline{DE}$ or $\overline{DE}$ = ½ $\overline{BE}$.

Dividing a Line Segment into Equal Parts

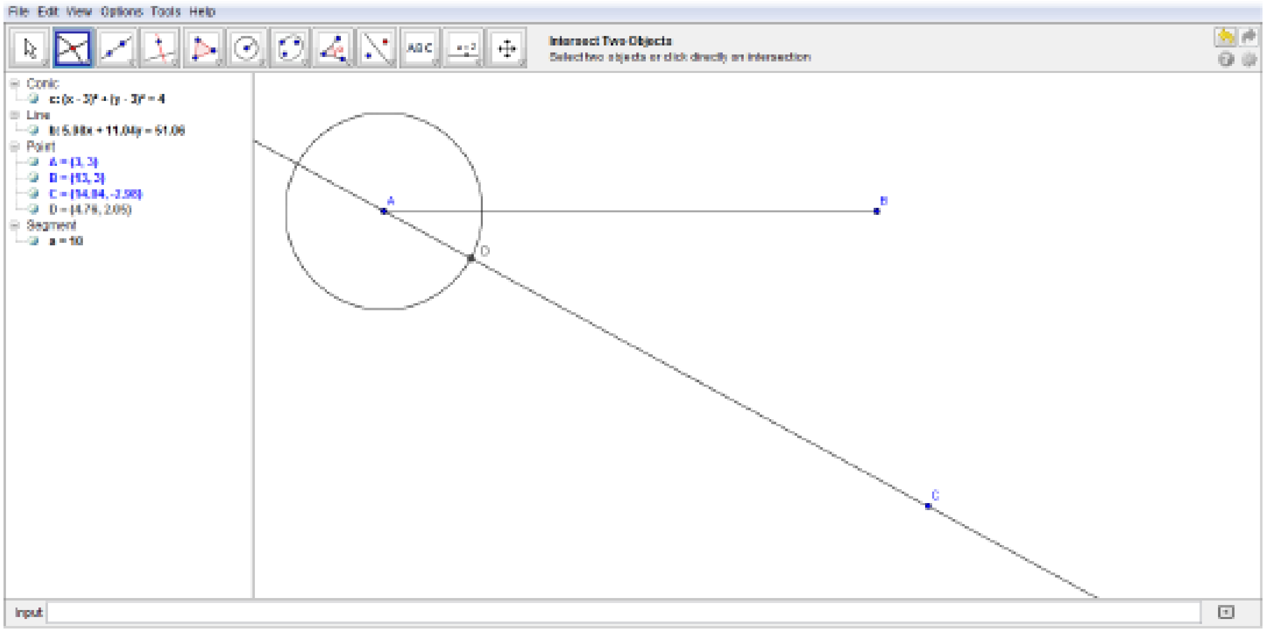
1. Create line segment $\overline{AB}$ that will be divided into a given number of equal parts, in this case $\overline{AB}$ will be divided into five equal parts. Use the 3rd option button and choose "Segment through Two Points" to create $\overline{AB}$.

2. Use the 3rd option button and choose "Line through Two Points" to create $\overleftrightarrow{AC}$.

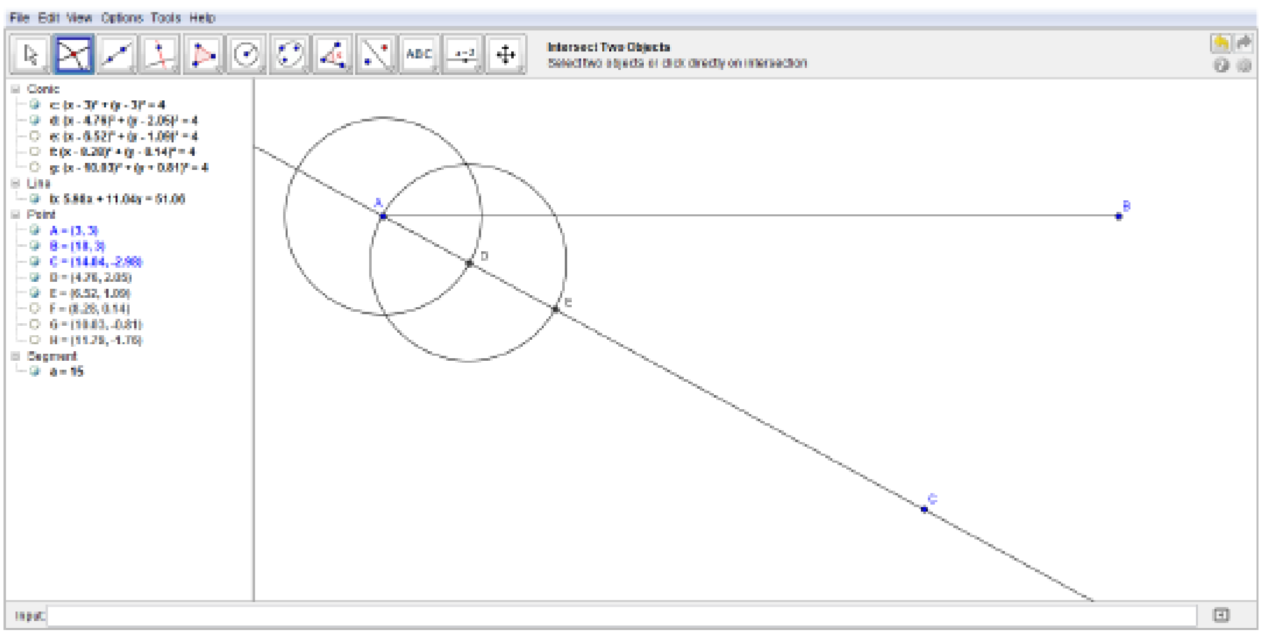


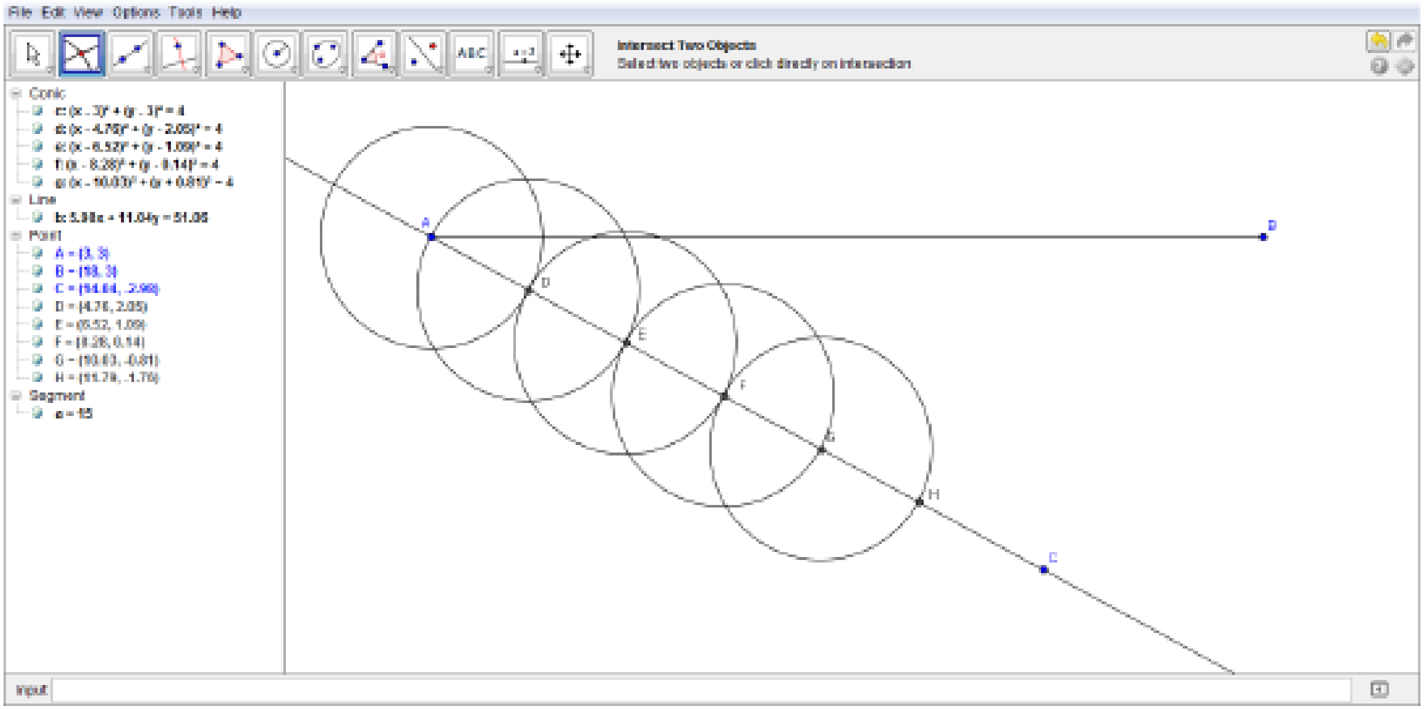
3. Use the 6th option button and choose "Circle with Center and Radius." Select point A as the center and enter a length for the radius. Record this radius length; this same length will be used throughout this construction.

4. Use the 2nd option button and choose "Intersect Two Objects" to locate point D, the intersection of Circle A and $\overleftrightarrow{AC}$.

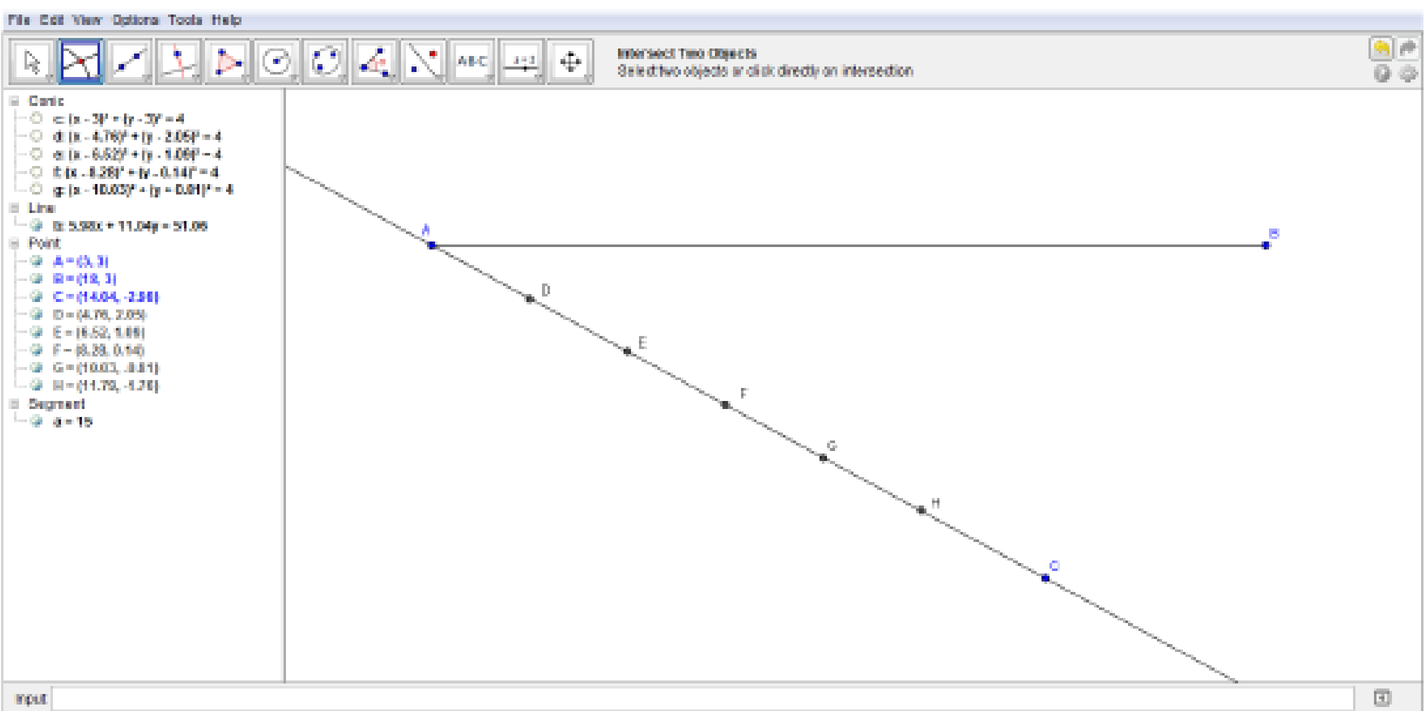


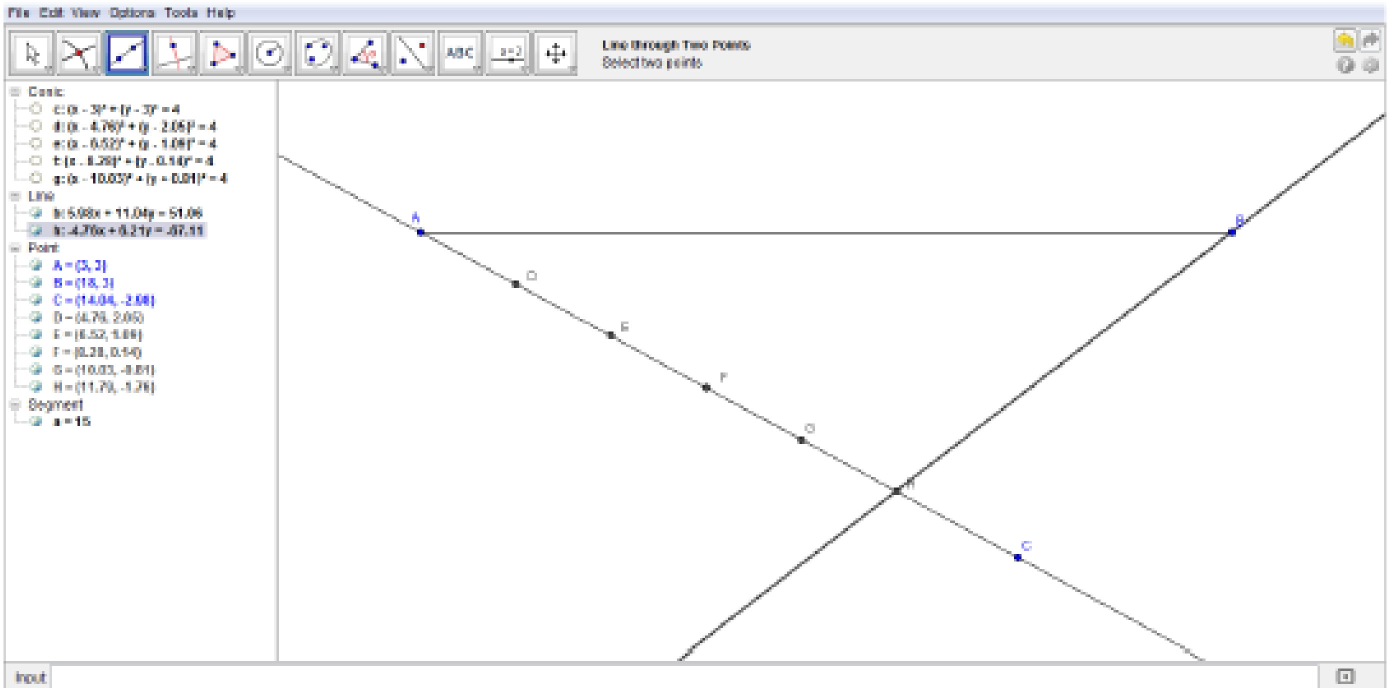
5. Repeat Steps 3 and 4, selecting point D as the center of the circle and entering the radius length used in Step 3. Locate the point of intersection (point E) of Circle D and $\overleftrightarrow{AC}$.





7. The equations of the circles are written in the left-hand column of the *GeoGebra* screen in the section labeled "Conic." Click on each of the bullets to hide the circles. ("Hiding" the circles will make it easier to see the rest of the construction.)

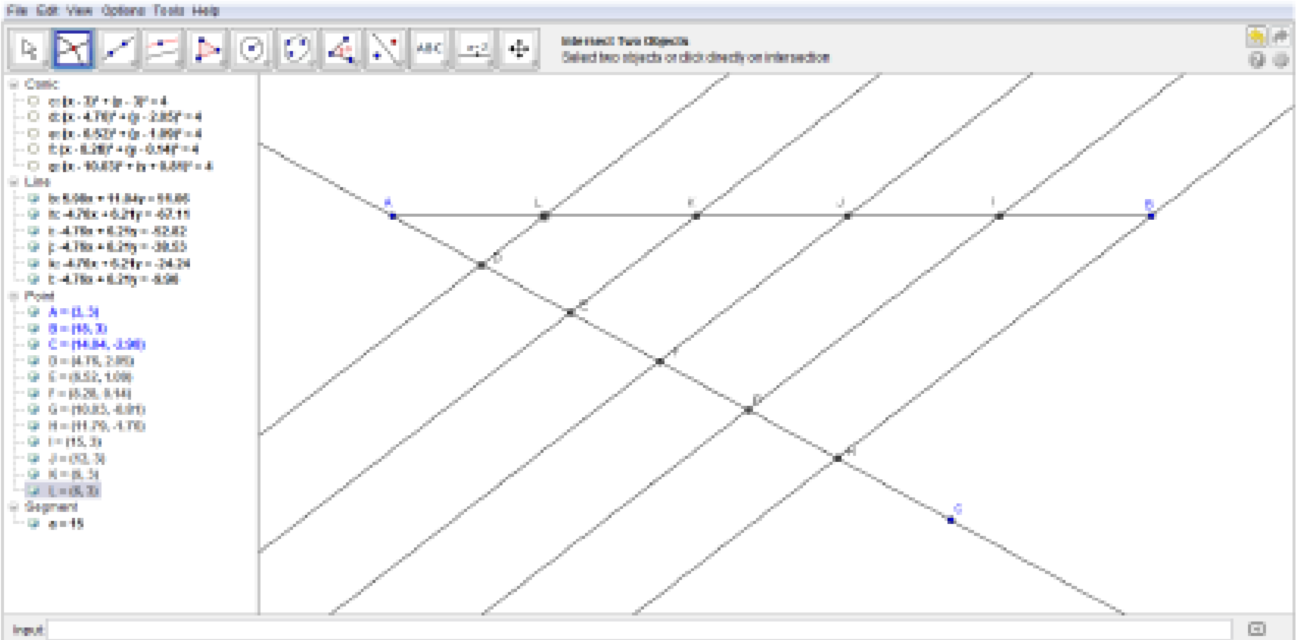




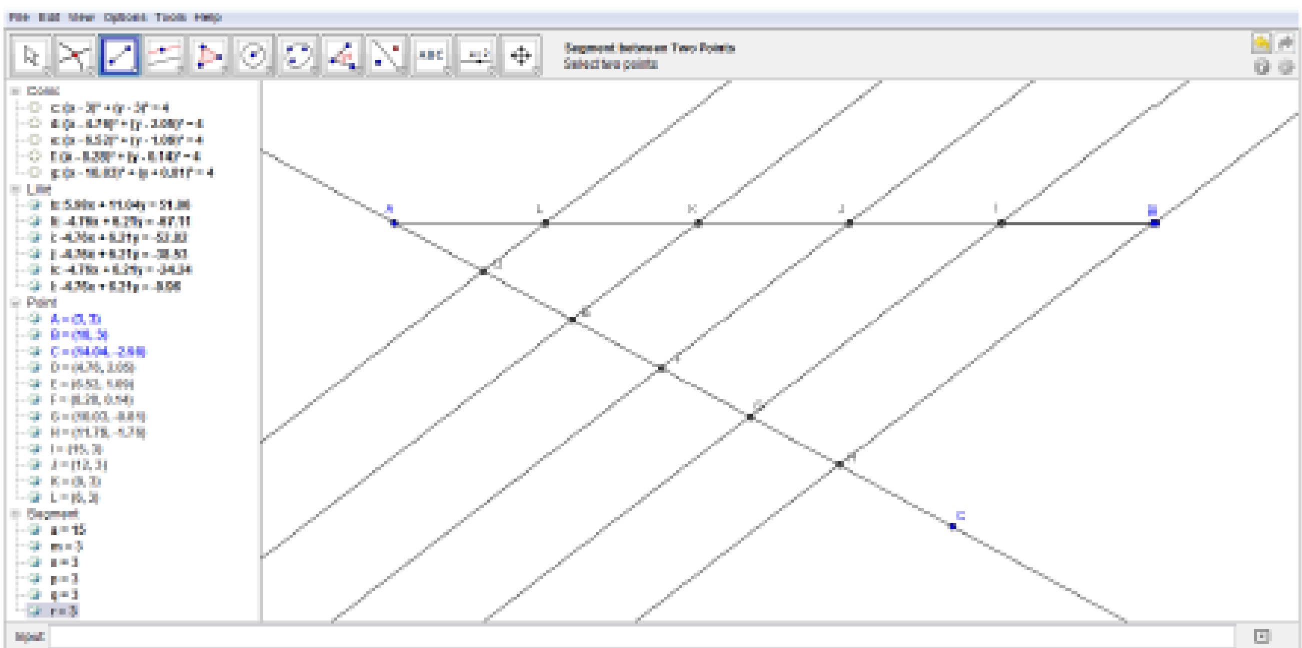
9. Use the 3rd option button and select "Parallel Line." Choose point G and $\overleftrightarrow{HB}$ to create a line through point G that is parallel to $\overleftrightarrow{HB}$.

10. Use the 2nd option button and choose "Intersect Two Objects" to locate point I, the point of intersection of $\overline{AB}$ and $\overleftrightarrow{GI}$.

11. Continue repeating the process of creating lines that are parallel to $\overleftrightarrow{HB}$ through points F, E, and D.

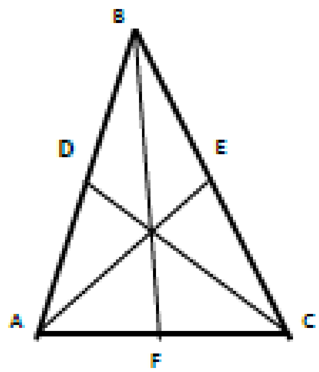


12. AB has now been divided into five equal parts. To verify this, use the 3rd option button and select "Segment between Two Points." As AL, LK, KL, JI, and IB are created, their lengths are displayed in the section "Segment" in the left-hand column. Note that these segments have equal length.



Triangle Medians

- On a sheet of patty paper, draw a large scalene triangle $\triangle ABC$.
- Locate the midpoints of segments $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. Label these midpoints D, E, and F.



- Create medians $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$.
- Use your compass to investigate whether there is anything special about the point of intersection of these three medians.
    - Is the point of intersection equidistant from the three vertices?
    - Is the point of intersection equidistant from the three sides?
    - Is the point of intersection equidistant from the midpoint of each median?

The point of intersection divides the median into two segments.

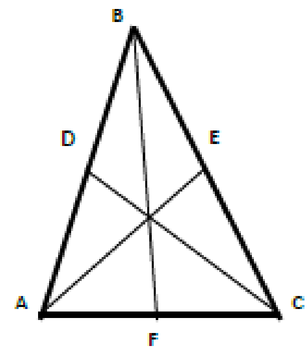
Use your compass to compare:

- the length of the shorter segment and the longer segment of median $\overline{AE}$
- the length of the shorter segment and the longer segment of median $\overline{BF}$
- the length of the shorter segment and the longer segment of median $\overline{CD}$

Complete the following statement:
The point of intersection of the three medians of a triangle divides each median into two parts so that the distance from the point of intersection to the vertex is ____________ the distance from the point of intersection to the midpoint of the opposite side.

Triangle Medians – Key

- *On a sheet of patty paper, draw a large scalene triangle △ABC.*
- Locate the midpoints of segments $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. Label these midpoints D, E, and F.



- Create medians $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$.
- Use your compass to investigate whether there is anything special about the point of intersection of these three medians.
  - Is the point of intersection equidistant from the three vertices?
  - Is the point of intersection equidistant from the three sides?
  - Is the point of intersection equidistant from the midpoint of each median?

The point of intersection divides the median into two segments.

Use your compass to compare:

- the length of the shorter segment and the longer segment of median $\overline{AE}$
- the length of the shorter segment and the longer segment of median $\overline{BF}$
- the length of the shorter segment and the longer segment of median $\overline{CD}$

Complete the following statement:
The point of intersection of the three medians of a triangle divides each median into two parts so that the distance from the point of intersection to the vertex is twice the distance from the point of intersection to the midpoint of the opposite side.

Investigation: Medians of a Triangle

QUESTION: What is the relationship between segments formed by the intersection of the medians of a triangle?

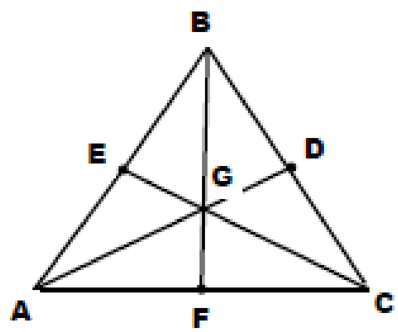
Explore:

1. **On a sheet of paper, draw any triangle and cut it out.**

2. **Find the midpoint of each side by folding each side, vertex to vertex, and pinching the paper at the middle.**

3. **Draw a segment from each midpoint to the vertex of the opposite angle. These segments are called *medians*.**

4. **Label the triangle as shown below. Complete the table.**



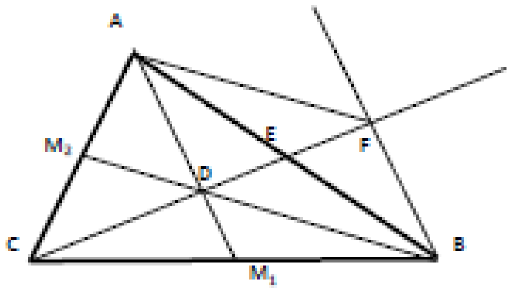
Length of median	**AD =**	**BF =**	**CE =**
Length of segment from G to vertex	**AG =**	**BG =**	**CG =**
Median length multiplied by $\frac{2}{3}$	$\frac{2}{3}$**AD =**	$\frac{2}{3}$**BF =**	$\frac{2}{3}$**CE =**

5. **Is there a relationship between the distance from G to a vertex and the distance from that vertex to the midpoint of the opposite side?**

Triangle Medians Proof

Prove: The medians of a triangle intersect at one point; and that point is 2/3 of the distance from any vertex to the midpoint of the opposite side.

- Let D be the point of intersection of the medians $\overline{AM_1}$ and $\overline{BM_2}$.
- Extend $\overline{CD}$ beyond its intersection with $\overline{AB}$ at E.
- Draw a line through B that is parallel to AM_1. Let F be its point of intersection with $\overrightarrow{CD}$.



- In ΔBCF, M_1 is the midpoint of $\overline{BC}$.
- $\overline{BF} || \overline{GM_1}$.
- Therefore, D is the midpoint of $\overline{CF}$.
- In ΔACF, M_2 is the midpoint of $\overline{AC}$ and D is the midpoint of $\overline{CF}$.
- Therefore, $\overline{AF} || \overline{DM_2}$, or $\overline{AF} || \overline{BD}$.
- With two pairs of parallel sides, the quadrilateral AFBD is a parallelogram.
- Since AFBD is a parallelogram, its diagonals bisect each other.
- Therefore, DE = EF and AE = EB.
- Therefore, E is the midpoint of $\overline{AB}$ and E is the midpoint of $\overline{DF}$.
- Since D is the midpoint of $\overline{CF}$, then CD = DE + EF
- Therefore, CD = 2DE
- Or, CD is 2/3 CE