West Virginia math4life

Educators' Guide for Mathematics

Algebra I

West Virginia DEPARTMENT OF EDUCATION

Algebra I

The main purpose of Algebra I is to develop students' fluency with linear, quadratic, and exponential functions. The critical areas of instruction involve deepening and extending students' understanding of linear and exponential relationships by comparing and contrasting those relationships and by applying linear models to data that exhibit a linear trend. In addition, students engage in methods for analyzing, solving, and using exponential and quadratic functions. Some of the overarching elements of the Algebra I course include the notion of *function*, solving equations, rates of change and growth patterns, graphs as representations of functions, and modeling.

For the Traditional Pathway, the standards in the Algebra I course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, and Statistics and Probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not simply topics to be checked off from a list during isolated units of instruction; rather, they represent content that should be present throughout the school year in rich instructional experiences.

What Students Learn in Algebra I

In Algebra I, students employ reasoning about structure to define and make sense of rational exponents and explore the algebraic structure of the rational and real number systems. They understand that numbers in real-world applications often have units attached to them—that is, the numbers are considered *quantities*. Students' work with numbers and operations throughout elementary and middle school has led them to an understanding of the structure of the number system; in Algebra I, students explore the structure of algebraic expressions and polynomials. They see that certain properties must persist when they work with expressions that are meant to represent numbers, which they now write in an abstract form involving variables. When two expressions with overlapping domains are set as equal to each other, resulting in an equation, an implied solution is set (be it empty or non-empty), and students not only refine their techniques for solving equations and finding the solution set, but they can clearly explain the algebraic steps they employed to do so.

Students begin their exploration of linear equations in middle school, first by connecting proportional equations ($y = kx$, $k \neq 0$) to graphs, tables, and real-world contexts, and then moving toward an understanding of general linear equations ($y = mx + b$, $m \neq 0$) and their graphs. In Algebra I, students extend this knowledge to work with absolute value equations, linear inequalities, and systems of linear equations. After learning a more precise definition of *function* in this course, students examine this new idea in the familiar context of linear equations—for example, by seeing the solution of a linear equation as solving $f(x) = g(x)$ for two linear functions f and g.

Students continue to build their understanding of functions beyond linear ones by investigating tables, graphs, and equations that build on previous understandings of numbers and expressions. They make connections between different representations of the same function. They also learn to build functions in a modeling context and solve problems related to the resulting functions. Note that in Algebra I the focus is on linear, simple exponential, and quadratic equations.

Finally, students extend their prior experiences with data, using more formal means of assessing how a model fits data. Students use regression techniques to describe approximately linear relationships between quantities. They use graphical representations and knowledge of the context to make judgments about the appropriateness of linear models wherein students examine residuals to analyze the goodness of fit.

Examples of Key Advances from Kindergarten through Grade Eight

- Having already extended arithmetic from whole numbers to fractions (grades four through six) and from fractions to rational numbers (grade seven), students in grade eight encounter specific irrational numbers such as $\sqrt{5}$ and π. In Algebra I, students begin to understand the real number system.
- Students in middle grades work with measurement units, including units obtained by multiplying and dividing quantities. In Algebra I (conceptual category Numbers and Quantity), students apply these skills in a more sophisticated fashion to solve problems in which reasoning about units adds insight.
- Algebraic themes beginning in middle school continue and deepen during high school. As early as grades six and seven, students begin to use the properties of operations to generate equivalent expressions (standards **M.6.14** and **M.7.7**). By grade seven, they begin to recognize that rewriting expressions in different forms could be useful in problem solving (standard **M.7.8**). In Algebra I, these aspects of algebra carry forward as students continue to use properties of operations to rewrite expressions, gaining fluency and engaging in what has been called "mindful manipulation."
- Students in grade eight extend their prior understanding of proportional relationships to begin working with functions, with an emphasis on linear functions. In Algebra I, students master linear and quadratic functions. Students encounter other kinds of functions to ensure that general principles of working with functions are perceived as applying to all functions, as well as to enrich the range of quantitative relationships considered in problems.
- Students in grade eight connect their knowledge about proportional relationships, lines, and linear equations (standards **M.8.7–M.8.8**). In Algebra I, students solidify their

understanding of the analytic geometry of lines. They understand that in the Cartesian coordinate plane:
    - the graph of any linear equation in two variables is a line;
    - any line is the graph of a linear equation in two variables.
- As students acquire mathematical tools from their study of algebra and functions, they apply these tools in statistical contexts (e.g., standard **M.A1HS.37**). In a modeling context, they might informally fit a quadratic function to a set of data, graphing the data and the model function on the same coordinate axes. They also draw on skills first learned in middle school to apply basic statistics and simple probability in a modeling context. For example, they might estimate a measure of center or variation and use it as an input for a rough calculation.
- Algebra I techniques open an extensive variety of solvable word problems that were previously inaccessible or very complex for students in kindergarten through grade eight. This expands problem solving dramatically.

Connecting Mathematical Habits of Mind and Content

The Mathematical Habits of Mind (**MHM**) apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, relevant, and meaningful subject. The MHM represent a picture of what it looks like for students to understand and do mathematics; therefore, to the extent possible, content instruction includes attention to appropriate practice standards. Students are afforded ample opportunities to engage in each Mathematical Habit of Mind in Algebra I. The following table offers some general examples.

Mathematical Habits of Mind Algebra I

Mathematical Habits of Mind	Explanation and Examples for Algebra I
MHM1 Make sense of problems and persevere in solving them.	Students learn that patience is often required to understand what a problem is asking. They discern between useful and extraneous information. They expand their repertoire of expressions and functions that can be used to solve problems.
MHM2 Reason abstractly and quantitatively.	Students extend their understanding of slope as the rate of change of a linear function to comprehend that the average rate of change of any function can be computed over an appropriate interval.
MHM3 Construct viable arguments and critique the reasoning of others.	Students reason through the solving of equations, recognizing that solving an equation involves more than simply following rote rules and steps. They use language such as "If ________, then ________" when explaining their solution methods and provide justification for their reasoning.
MHM4 Model with mathematics.	Students also discover mathematics through experimentation and examination of data patterns from real-world contexts. Students apply their new mathematical understanding of exponential, linear, and quadratic functions to real-world problems.
MHM5 Use appropriate tools strategically.	Students develop a general understanding of the graph of an equation or function as a representation of that object, and they use tools such as graphing calculators or graphing software to create graphs in more complex examples to interpret results. They also construct diagrams to solve problems.
MHM6 Attend to precision.	Students begin to understand that a *rational number* has a specific definition and that *irrational numbers* exist. They make use of the definition of *function* when deciding if an equation can describe a function by asking, "Does every input value have exactly one output value?"
MHM7 Look for and make use of structure.	Students develop formulas such as $(a \pm b)^2 = a^2 \pm 2ab + b^2$ by applying the distributive property. Students see that the expression $5 + (x - 2)^2$ takes the form of 5 plus "something squared," and because "something squared" must be positive or zero, the expression can be no smaller than 5.
MHM8	Students see that the key feature of a line in the plane is an

Look for and express regularity in repeated reasoning.	equal difference in outputs over equal intervals of inputs, and that the result of evaluating the expression $\frac{y_2-y_1}{x_2-x_1}$ for points on the line is always equal to a certain number m. Therefore, if (x, y) is a generic point on this line, the equation $m = \frac{y-y_1}{x-x_1}$ will give a general equation of that line.

MHM4 holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. In the description of the Algebra I content standards that follow, Modeling is covered first to emphasize its importance in the higher mathematics curriculum. Examples where specific Mathematical Habits of Mind can be implemented in the Algebra I standards are noted in parentheses, with the standard(s) also listed.

Algebra I Content Standards, by Conceptual Category

The Algebra I course is organized by domains, clusters, and then standards. The overall purpose and progression of the standards included in Algebra I are described below, according to each conceptual category. Standards that are considered new for teachers of secondary grades are discussed more thoroughly than other standards.

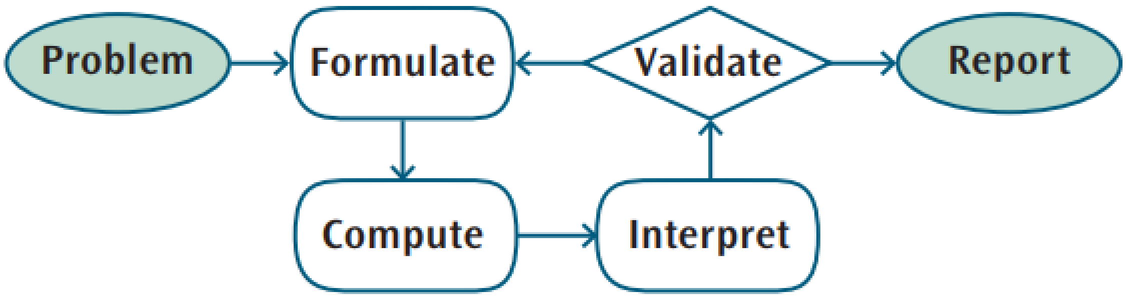
Conceptual Category: Modeling

Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them, and the mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise: Which of the quantities present in this situation are known, and which are unknown? Can a table of data be made? Is there a functional relationship in this situation? Students need to decide on a solution path, which may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They use previously derived models (e.g., linear functions), but may find that answering their question requires solving an equation and knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, etc.), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out;

see the following figure. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Modeling Cycle



The examples in this chapter are framed as much as possible to illustrate the concept of mathematical modeling. The important ideas surrounding linear and exponential functions, graphing, solving equations, and rates of change are explored through this lens.

Conceptual Category: Functions

Functions describe situations where one quantity determines another. For example, the return on $10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because we continually form theories about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models. In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, v; the rule $T(v) = \frac{100}{v}$ expresses this relationship algebraically and defines a function whose name is T.

The set of inputs to a function is called its *domain*. We often assume the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context. When describing relationships between quantities, the defining characteristic of a function is that the input value determines the output value, or equivalently, that the output value depends upon the input value (University of Arizona [UA] Progressions Documents 2013c, 2).

A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, “I’ll give you a state, you give me the capital city”; by an assignment, such as the fact that each individual is given a unique Social Security Number; by an algebraic expression, such as $f(x) = a + bx$; or by a recursive rule, such as $f(n+1) = f(n) + b$, $f(0) = a$. The graph of a function is often a useful way of visualizing the relationship that the function models, and manipulating a mathematical expression for a function can shed light on the function’s properties.

Linear and Exponential Relationships

Understand the concept of a function and use function notation.

M.A1HS.18

Recognize that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then f(x) denotes the output of f corresponding to the input x. The graph of f is the graph of the equation y = f(x). Instructional Note: Students should experience a variety of types of situations modeled by functions. Detailed analysis of any particular class of function at this stage is not advised. Students should apply these concepts throughout their future mathematics courses. Draw examples from linear functions and exponential functions having integral domains.

M.A1HS.19

Use function notation, evaluate functions for inputs in their domains and interpret statements that use function notation in terms of a context. Instructional Note: Students should experience a variety of types of situations modeled by functions. Detailed analysis of any particular class of function at this stage is not advised. Students should apply these concepts throughout their future mathematics courses. Draw examples from linear functions and exponential functions having integral domains.

M.A1HS.20

Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. (e.g., The Fibonacci sequence is defined recursively by f(0) = f(1) = 1, f(n+1) = f(n)+ f(n-1) for n ≥ 1. Instructional Note: Students should experience a variety of types of situations modeled by functions. Detailed analysis of any particular class of function at this stage is not advised. Students should apply these concepts throughout their future mathematics courses. Draw examples from linear functions and exponential functions having integral domains. Draw connection to M.A1HS.27, which requires students to write arithmetic and geometric sequences. Emphasize arithmetic and geometric sequences as examples of linear and exponential functions.

Interpret functions that arise in applications in terms of a context.

M.A1HS.21

For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. Instructional Note: Focus on linear and exponential functions.

M.A1HS.22

Relate the domain of a function to its graph and where applicable, to the quantitative relationship it describes. (e.g., If the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.) Instructional Note: Focus on linear and exponential functions.

M.A1HS.23

Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. Instructional Note: Focus on linear functions and exponential functions whose domain is a subset of the integers. The Unit on Quadratic Functions and Modeling in this course and the Algebra II course address other types of functions.

Quadratic Functions and Modeling
Interpret functions that arise in applications in terms of a context.
M.A1HS.51 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. Instructional Note: Focus on quadratic functions; compare with linear and exponential functions studied in the Unit on Linear and Exponential Relationships.
M.A1HS.52 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function. Instructional Note: Focus on quadratic functions; compare with linear and exponential functions studied in the Unit on Linear and Exponential Relationships.
M.A1HS.53 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. Instructional Note: Focus on quadratic functions; compare with linear and exponential functions studied in the Unit on Linear and Exponential Relationships.

While the grade-eight standards call for students to work informally with functions, students in Algebra I begin to refine their understanding and use the formal mathematical language of functions. Standards **M.A1HS.18 - M.A1HS.25, M.A1HS.51– M.A1HS.53** deal with understanding the concept of a function, interpreting characteristics of functions in context, and representing functions in different ways (**MHM6**). In **M.A1HS.18– M.A1HS.20**, students learn the language of functions and that a function has a domain that must be specified as well as a corresponding range. For instance, the function f where $f(n) = 4(n-2)^2$, defined for n, an integer, is a different function than the function g where $g(x) = 4(x-2)^2$ and g is defined for all real numbers x. Students make the connection between the graph of the equation $y = f(x)$ and the function itself—namely, that the coordinates of any point on the graph represent an input and output, expressed as $(x, f(x))$, and understand that the graph is a *representation* of a function. They connect the domain and range of a function to its graph (**M.A1HS.22**). Note that there is neither an exploration of the notion of *relation* vs. *function* nor the *vertical line test* in the West Virginia College- and Career- Readiness Standards. This is by design. The core question when investigating functions is, “Does each element of the domain correspond to exactly one element of the range?” (UA Progressions Documents 2013c, 8).

Standard **M.A1HS.20** represents a topic that is new to the traditional Algebra I course: *sequences*. Sequences are functions with a domain consisting of a subset of the integers. In grades four and five, students begin to explore number patterns, and this work leads to a full progression of ratios and proportional relationships in grades six and seven. Patterns are examples of sequences, and the work here is intended to formalize and extend students’ earlier understandings. For a simple example, consider the sequence 4, 7, 10, 13, 16 . . . , which might be described as a “plus 3 pattern” because terms are computed by adding 3 to the previous term. If we decided that 4 is the first term of the sequence, then we can make a table, a graph, and eventually a recursive rule for this sequence: $f(1) = 4$, $f(n+1) = f(n) + 3$ for $n \geq 2$. Of course, this sequence can also be described with the explicit formula $f(n) = 3n + 1$ for $n \geq 1$. Notice that the domain is included in the description of the rule (adapted from UA Progressions Documents 2013c, 8). In Algebra I, students should have opportunities to work with linear, quadratic, and exponential sequences and to interpret the parameters of the expressions defining the terms of the sequence when they arise in context.

Linear and Exponential Relationships
Analyze functions using different representations. **M.A1HS.24** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. a. Graph linear and quadratic functions and show intercepts, maxima, and minima. b. Graph exponential and logarithmic functions, showing intercepts and end behavior and trigonometric functions, showing period, midline, and amplitude. Instructional Note: Focus on linear and exponential functions. Include comparisons of two functions presented algebraically. For example, compare the growth of two linear functions, or two exponential functions such as $y = 3^n$ and $y = 100^{2n}$) **M.A1HS.25** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). (e.g., Given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.) Instructional Note: Focus on linear and exponential functions. Include comparisons of two functions presented algebraically. For example, compare the growth of two linear functions, or two exponential functions such as $y = 3^n$ and $y = 100^{2n}$).

Quadratic Functions and Modeling
Analyze functions using different representations. **M.A1HS.54** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. a. Graph linear and quadratic functions and show intercepts, maxima, and minima. b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. Instructional Note: Compare and contrast absolute value, step and piecewise-defined functions with linear, quadratic, and exponential functions. Highlight issues of domain, range, and usefulness when examining piecewise-defined functions. Extend work with quadratics to include the relationship between coefficients and roots, and that once roots are known, a quadratic equation can be factored. **M.A1HS.55** Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. b. Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, and classify them as representing exponential growth or decay. Instructional Note: Extend work with quadratics to include the relationship between coefficients and roots, and that once roots are known, a quadratic equation can be factored. This standard extends the work begun in the Linear and Exponential Relationships unit on exponential functions with integer exponents. **M.A1HS.56** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum. Instructional Note: Highlight issues of domain, range, and usefulness when examining piecewise-defined functions. Extend work with quadratics to include the relationship between coefficients and roots, and that once roots are known, a quadratic equation can be factored.

In standards **M.A1HS.24–25, M.A1HS.54– M.A1HS.56**, students represent functions with graphs and identify key features in the graph. In Algebra I, linear, exponential, and quadratic functions are given extensive treatment because they have their own group of standards dedicated to them. Students are expected to develop fluency only with linear, exponential, and quadratic functions in Algebra I, which includes the ability to graph them by hand.

In this set of three standards, students represent the same function algebraically in different forms and interpret these differences in terms of the graph or context. For instance, students may easily see that the graph of the equation $f(x) = 3x^2 + 9x + 6$ crosses the y-axis at $(0,6)$, since the terms containing x are simply 0 when $x = 0$—but then they factor the expression defining f to obtain $f(x) = 3(x + 2)(x + 1)$, easily revealing that the function crosses the x-axis at $(-2,0)$ and $(-1,0)$, since this is where $f(x) = 0$ (**MHM7**).

Linear and Exponential Relationships

Build a function that models a relationship between two quantities.

M.A1HS.26

Write a function that describes a relationship between two quantities.

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.
b. Combine standard function types using arithmetic operations. (e.g., Build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.)

Instructional Note: Limit to linear and exponential functions.

M.A1HS.27

Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. Instructional Note: Limit to linear and exponential functions. Connect arithmetic sequences to linear functions and geometric sequences to exponential functions.

Build new functions from existing functions.

M.A1HS.28

Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. Instructional Note: Focus on vertical translations of graphs of linear and exponential functions. Relate the vertical translation of a linear function to its y-intercept. While applying other transformations to a linear graph is appropriate at this level, it may be difficult for students to identify or distinguish between the effects of the other transformations included in this standard.

Quadratic Functions and Modeling

Build a function that models a relationship between two quantities.

M.A1HS.57

Write a function that describes a relationship between two quantities.

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.
b. Combine standard function types using arithmetic operations. For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.

Instructional Note: Focus on situations that exhibit a quadratic relationship.

Build new functions from existing functions.

M.A1HS.58

Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. Instructional Note: Focus on quadratic functions, and consider including absolute value functions.

M.A1HS.59

Find inverse functions. Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. For example, $f(x) = 2x^3$ or f(x) = (x+1)/(x-1) for x ≠ 1. Instructional Note: Focus on linear functions but consider simple situations where the domain of the function must be restricted in order for the inverse to exist, such as $f(x) = x^2$, x > 0.

Knowledge of functions and expressions is only part of the complete picture. One must be able to understand a given situation and apply function reasoning to model how quantities change together. Often, the function created sheds light on the situation at hand; one can make predictions of future changes, for example. This is the content of standards **M.A1HS.26, M.A1HS.57** and **M.A1HS.27.** A strong connection exists between standards **M.A1HS.26, M.A1HS.57** and standard **M.A1HS.46**, which discusses creating equations. The following example shows that students can create functions based on prototypical ones.

Example: Exponential Growth	**M.A1HS.26, M.A1HS.27, M.A1HS.57**

When a quantity grows with time by a multiplicative factor greater than 1, it is said the quantity grows exponentially. Hence, if an initial population of bacteria, P_0, doubles each day, then after t days, the new population is given by $P(t) = P_0 2^t$. This expression can be generalized to include different growth rates, r, as in $P(t) = P_0 r^t$. A more specific example illustrates the type of problem that students may face after they have worked with basic exponential functions:

On June 1, a fast-growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If the algae continue to grow unabated, the lake will be totally covered, and the fish in the lake will suffocate. Based on the current rate at which the algae are growing, this will happen on June 30.

Possible Questions to Ask:

a. When will the lake be covered halfway?
b. b. Write an equation that represents the percentage of the surface area of the lake that is covered in algae, as a function of time (in days) that passes since the algae were introduced into the lake.

Solution and Comments:

a. Since the population doubles each day, and since the entire lake will be covered by June 30, this implies that half the lake was covered on June 29.
b. If $P(t)$ represents the *percentage* of the lake covered by the algae, then we know that $P(29) = P_0 2^{29} = 100$ (note that June 30 corresponds to $t = 29$). Therefore, we can solve for the initial percentage of the lake covered, $P_0 = \frac{100}{2^{29}} = 1.86 \times 10^{-7}$. The equation for the percentage of the lake covered by algae at time t is therefore $P(t) = (1.86 \times 10^{-7})2^t$.

Adapted from Illustrative Mathematics 2013i.

As mentioned earlier, the study of arithmetic and geometric sequences, written both explicitly and recursively (**M.A1HS.27**), is new to the Algebra I course. When presented with a sequence, students can often manage to find the recursive pattern of the sequence (i.e., how the sequence changes from term to term). For instance, a simple doubling pattern can lead to an exponential expression of the form $a2^n$, for $n \geq 0$. Ample experience with linear and exponential functions—which show equal differences over equal intervals and equal ratios over intervals, respectively—can provide students with tools for finding explicit rules for sequences. Investigating the simple sequence of squares, $f(x) = n^2$ (where $n \geq 1$), provides a prototype for other basic quadratic sequences. Diagrams, tables, and graphs can help students make sense of the different rates of growth all three sequences exhibit.

Example: Exponential Growth	**M.A1HS.27**

Cellular Growth

Populations of cyanobacteria can double every 6 hours under ideal conditions, at least until the nutrients in its supporting culture are depleted. This means a population of 500 such bacteria would grow to 1000 in the first 6-hour period, 2000 in the second 6-hour period, 4000 in the third 6-hour period, and so on. Evidently, if n represents the number of 6-hour periods from the start, the population at that time $P(n)$ satisfies $P(n) = 2 \bullet P(n - 1)$. This is a *recursive* formula for the sequence $P(n)$, which gives the population at a given time period n in terms of the population at time period $n - 1$. To find a closed, *explicit*, formula for $P(n)$, students can reason that

$$P(0) = 500, P(1) = 2 \bullet 500, P(2) = 2 \bullet 2 \bullet 500, P(3) = 2 \bullet 2 \bullet 2 \bullet 500, \ldots$$

A pattern emerges: that $P(n) = 2^n \bullet 500$. In general, if an initial population P_0 grows by a factor $r > 1$ over a n fixed time period, then the population after n time periods is given by $P(n) = P_0 r^n$.

The content of standards **M.A1HS.28** and **M.A1HS.58** has typically been left to later courses. In Algebra I, the focus is on linear, exponential, and quadratic functions. Even and odd functions are addressed in later courses. In keeping with the theme of the input–output interpretation of a function, students develop an understanding of the effect on the output of a function under certain transformations, such as in the table below:

Expression	**Interpretation**
$f(a + 2)$	The output when the input is 2 greater than a
$f(a) + 3$	3 more than the output when the input is a
$2f(x) + 5$	5 more than twice the output of f when the input is x

Such understandings can help students to see the effect of transformations on the graph of a function, and in particular, can aid in understanding why it appears that the effect on the graph is the opposite to the transformation on the variable. For example, the graph of $y = f(x + 2)$ is the graph of f shifted 2 units to the left, not to the right (UA Progressions Documents 2013c, 7).

Also new to the Algebra I course is standard **M.A1HS.59**, which calls for students to find inverse functions 9 in simple cases. For example, an Algebra I student might solve the equation $F = \frac{9}{5}C + 32$ for C. The student starts with this formula, showing how Fahrenheit temperature is a function of Celsius temperature, and by solving for C, finds the formula for the inverse function. This is a contextually appropriate way to find the expression for an inverse function, in contrast with the practice of simply swapping x and y in an equation and solving for y.

Linear and Exponential Relationships

Construct and compare linear, quadratic, and exponential models and solve problems.

M.A1HS.29

Distinguish between situations that can be modeled with linear functions and with exponential functions.

a. Prove that linear functions grow by equal differences over equal intervals; exponential functions grow by equal factors over equal intervals.
b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.
c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

M.A1HS.30

Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship or two input-output pairs (include reading these from a table). Instructional Note: In constructing linear functions, draw on and consolidate previous work in Grade 8 on finding equations for lines and linear functions.

M.A1HS.31

Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. Instructional Note: Limit to comparisons between exponential and linear models.

Interpret expressions for functions in terms of the situation they model.

M.A1HS.32

Interpret the parameters in a linear or exponential function in terms of a context. Instructional Note: Limit exponential functions to those of the form $f(x) = b^x + k$.

Quadratic Functions and Modeling

Construct and compare linear, quadratic and exponential models and solve problems.
M.A1HS.60
Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. Instructional Note: Compare linear and exponential growth to quadratic growth.

Modeling the world often involves investigating rates of change and patterns of growth. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. In standards **M.A1HS.29**, students recognize and understand the defining characteristics of linear, quadratic, and exponential functions. Students have already worked extensively with linear equations. They have developed an understanding that an equation in two variables of the form $y = mx + b$ exhibits a special relationship between the variables x and y—namely, that a change of Δx in the variable x, the independent variable, results in a change of $\Delta y = m \bullet \Delta x$ in the dependent variable y. They have seen this informally, in graphs and tables of linear relationships, starting in the grade-eight standards (**M.8.7, M.8.8, M.8.13**). For example, students recognize that for successive whole-number input values, x and $x + 1$, a linear function f, defined by $f(x) = mx + b$, exhibits a constant rate of change:

$$f(x+1) - f(x) = [m(x+1) + b] - (mx + b) = m(x + 1 - x) = m$$

Standard **M.A1HS.29a** requires students to prove that linear functions exhibit such growth patterns.

In contrast, an exponential function exhibits a constant percent change in the sense that such functions exhibit a constant *ratio* between output values for successive input values.[1] For instance, a t-table for the equation $y = 3^x$ illustrates the constant ratio of successive y-values for this equation:

n	$y = 3^n$	**Ratio of successive y-values**
1	3	
2	9	$\frac{9}{3} = 3$
3	27	$\frac{27}{9} = 3$
4	81	$\frac{81}{27} = 3$

1. In Algebra I of the West Virginia College- and Career- Readiness Standards for Mathematics, only integer values for x are considered in exponential equations such as $y = b^x$.

In general, a function g, defined by $g(x) = ab^x$, can be shown to exhibit this constant ratio growth pattern:

$$\frac{g(x+1)}{g(x)} = \frac{ab^{x+1}}{ab^x} = \frac{b^{x+1}}{b^x} = b^{(x+1)-x} = b$$

In Algebra I, students are not required to prove that exponential functions exhibit this growth rate; however, they must be able to recognize situations that represent both linear and exponential functions and construct functions to describe the situations (**M.A1HS.30**). Finally, students interpret the *parameters* in linear, exponential, and quadratic expressions and model physical problems with such functions. The meaning of parameters often becomes much clearer when they are presented in a modeling situation rather than in an abstract way.

A graphing utility, spreadsheet, or computer algebra system can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions (**MHM5**). Real-world examples where this can be explored involve half-lives of pharmaceuticals, investments, mortgages, and other financial instruments. For example, students can develop formulas for annual compound interest based on a general formula, such as $P(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}$, where r is the interest rate, n is the number of times the interest is compounded per year, and t is the number of years the money is invested. They can explore values after different time periods and compare different rates and plans using computer algebra software or simple spreadsheets (**MHM5**). This hands-on experimentation with such functions helps students develop an understanding of the functions' behavior.

Conceptual Category: Number and Quantity

In the grade-eight standards, students encounter some examples of irrational numbers, such as π and $\sqrt{2}$ (or $\sqrt{n}$ where n is a non-square number). In Algebra I, students extend this understanding beyond the fact that there are numbers that are not rational; they begin to understand that the rational numbers form a closed system. Students witness that with each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—whole numbers, rational numbers, and real numbers—the distributive law continues to hold, and the commutative and associative laws are still valid for both addition and multiplication. However, in Algebra I students go further along this path.

Linear and Exponential Relationships
Extend the properties of exponents to rational exponents.
M.A1HS.11

Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. (e.g., We define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.) Instructional Note: Address this standard before discussing exponential functions with continuous domains.

M.A1HS.12
Rewrite expressions involving radicals and rational exponents using the properties of exponents. Instructional Note: Address this standard before discussing exponential functions with continuous domains.

Quadratic Functions and Modeling

Use properties of rational and irrational numbers.

M.A1HS.50
Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational. Instructional Note: Connect to physical situations (e.g., finding the perimeter of a square of area 2).

With **M.A1HS.11**, students make meaning of the representation of radicals with rational exponents. Students are first introduced to exponents in grade six; by the time they reach Algebra I, they should have an understanding of the basic properties of exponents (e.g., that $x^n \bullet x^m = x^{n+m}$, $(x^n)^m = x^{nm}$, $\frac{x^n}{x^m} = x^{n-m}$, $x^0 = 1$ for $x \neq 0$,). In fact, they may have justified certain properties of exponents by reasoning with other properties (**MHM3, MHM7**), for example, justifying why any non-zero number to the power 0 is equal to 1:

$$x^0 = x^{n-n} = \frac{x^n}{x^n} = 1, for\ x \neq 0$$

They further their understanding of exponents in Algebra I by using these properties to explain the meaning of rational exponents. For example, properties of whole-number exponents suggest that $\left(5^{1/3}\right)^3$ should be the same as $5^{[(1/3)\bullet 3]} = 5^1 = 5$, so that $5^{1/3}$ should represent the cube root of 5 . In addition, $(ab)^n = a^n \bullet b^n$ reveals that $\sqrt{20} = (4 \bullet 5)^{1/2} = 4^{1/2} \bullet 5^{1/2} = \sqrt{4} \bullet \sqrt{5} = 2\sqrt{5}$, which shows that $\sqrt{20} = 2\sqrt{5}$. The intermediate steps of writing the square root as a rational exponent are necessary at first, but eventually students can work more quickly, understanding the reasoning underpinning this process. Students extend such work with radicals and rational exponents to variable expressions as well—for example, rewriting an expression like $(a^2b^5)^{3/2}$ using radicals (**M.A1HS.12**).

In standard **M.A1HS.50**, students explain that the sum or product of two rational numbers is rational, arguing that the sum of two fractions with integer numerator and denominator is also a fraction of the same type, which shows that the rational numbers are *closed* under the operations of addition and multiplication (**MHM3**). The notion that this set of numbers is closed under these operations will be extended to the sets of polynomials and rational functions in later courses. Moreover, students argue that the sum of a rational and an irrational is irrational, and the product of a non-zero rational and an irrational is still irrational, showing that the irrational numbers are truly another unique set of numbers that, along with the rational numbers, forms a larger system, the *system* of real numbers (**MHM3, MHM7**).

Relationships between Quantities and Reasoning with Equations
Reason quantitatively and use units to solve problems. **M.A1HS.1** Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. **M.A1HS.2** Define appropriate quantities for the purpose of descriptive modeling. Instructional Note: Working with quantities and the relationships between them provides grounding for work with expressions, equations, and functions. **M.A1HS.3** Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

In real-world problems, the answers are usually not pure numbers, but *quantities*: numbers with units, which involve measurement. In their work in measurement up through grade eight, students primarily measure commonly used attributes such as length, area, and volume. In higher mathematics, students encounter a wider variety of units in modeling—for example, when considering acceleration, currency conversions, derived quantities such as person-hours and heating degree-days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages.

In Algebra I, students use units to understand problems and make sense of the answers they deduce. The following example illustrates the facility with units that students are expected to attain in this domain.

Example **M.A1HS.1– M.A1HS.3**

As Felicia gets on the freeway to drive to her cousin's house, she notices that she is a little low on fuel. There is a gas station at the exit she normally takes, and she wonders if she will need gas before reaching that exit. She normally sets her cruise control at the speed limit of 70 mph, and the freeway portion of the drive takes about an hour and 15 minutes. Her car gets about 30 miles per gallon on the freeway. Gas costs $3.50 per gallon.

a. Describe an estimate that Felicia might form in her head while driving to decide how many gallons of gas she needs to make it to the gas station at her usual exit.

b. Assuming she makes it, how much does Felicia spend per mile on the freeway?

Solution:

a. To estimate the amount of gas she needs, Felicia calculates the distance traveled at 70 mph for 1.25 hours. She might calculate as follows:

$$70 \bullet 1.25 = 70 + (0.25 \bullet 70) = 70 + 17.5 = 87.5\ miles$$

Since 1 gallon of gas will take her 30 miles, 3 gallons of gas will take her 90 miles—a little more than she needs. So she might figure that 3 gallons is enough.

b. Since Felicia pays $3.50 for one gallon of gas, and one gallon of gas takes her 30 miles, it costs her $3.50 to travel 30 miles. Therefore:

$$\frac{\$3.50}{30\ miles} \approx \frac{\$0.12}{1\ mile}$$

Hence, it costs Felicia 12 cents to travel each mile on the freeway.

Adapted from Illustrative Mathematics 2013o.

Conceptual Category: Algebra

In the Algebra conceptual category, students extend the work with expressions that they started in the middle-grades standards. They create and solve equations in context, utilizing the power of variable expressions to model real-world problems and solve them with attention to units and the meaning of the answers they obtain. They continue to graph equations, understanding the resulting picture as a representation of the points satisfying the equation. This conceptual category accounts for a large portion of the Algebra I course and, along with the Functions category, represents the main body of content.

The Algebra conceptual category in higher mathematics is very closely related to the Functions conceptual category (UA Progressions Documents 2013b, 2):

- An expression in one variable can be viewed as defining a function: the act of evaluating the expression is an act of producing the function's output given the input.
- An equation in two variables can sometimes be viewed as defining a function, if one of the variables is designated as the input variable and the other as the output variable, and if there is just one output for each input. This is the case if the expression is of the form $y = (expression\ in\ x)$ or if it can be put into that form by solving for y.
- The notion of equivalent expressions can be understood in terms of functions: if two expressions are equivalent, they define the same function.
- The solutions to an equation in one variable can be understood as the input values that yield the same output in the two functions defined by the expressions on each side of the equation. This insight allows for the method of finding approximate solutions by graphing functions defined by each side and finding the points where the graphs intersect.

Thus, in light of understanding functions, the main content of the Algebra category (solving equations, working with expressions, and so forth) has a very important purpose.

Relationships between Quantities and Reasoning with Equations

Interpret the structure of expressions.

M.A1HS.4

Interpret expressions that represent a quantity in terms of its context.

a. Interpret parts of an expression, such as terms, factors, and coefficients.
b. Interpret complicated expressions by viewing one or more of their parts as a single entity. (e.g., Interpret P(1 + r)n as the product of P and a factor not depending on P. Instructional Note: Limit to linear expressions and to exponential expressions with integer exponents.

Expressions and Equations

Interpret the structure of equations.

M.A1HS.41

Interpret expressions that represent a quantity in terms of its context.

a. Interpret parts of an expression, such as terms, factors, and coefficients.
b. Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret P(1 + r)n as the product of P and a factor not depending on P. Instructional Note: Exponents are extended from the integer exponents found in the unit on Relationships between Quantities and Reasoning with Equations to rational exponents focusing on those that

represent square or cube roots.
Instructional Note: Focus on quadratic and exponential expressions.

M.A1HS.42
Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. Instructional Note: Focus on quadratic and exponential expressions.

An expression can be viewed as a recipe for a calculation, with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price, p. Rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor. Students begin this work in grades six and seven and continue this work with more complex expressions in Algebra I.

Expressions and Equations

Write expressions in equivalent forms to solve problems.

M.A1HS.43
Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

a. Factor a quadratic expression to reveal the zeros of the function it defines.
b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
c. Use the properties of exponents to transform expressions for exponential functions. For example the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.

Instructional Note: It is important to balance conceptual understanding and procedural fluency in work with equivalent expressions. For example, development of skill in factoring and completing the square goes hand-in-hand with understanding what different forms of a quadratic expression reveal.

In Algebra I, students work with examples of more complicated expressions, such as those that involve multiple variables and exponents. Students use the distributive property to investigate equivalent forms of quadratic expressions—for example, by writing

$$\begin{aligned}(x+y)(x-y) &= x(x-y)+y(x-y)\\ &= x^2-xy+xy-y^2\\ &= x^2-y^2\end{aligned}$$

This yields a special case of a factorable quadratic, the difference of squares.

Students factor second-degree polynomials by making use of such special forms and by using factoring techniques based on properties of operations (**M.A1HS.42**). Note that the standards avoid talking about "simplification," because the simplest form of an expression is often unclear, and even in cases where it is clear, it is not obvious that the simplest form is desirable for a given purpose. The standards emphasize purposeful transformation of expressions into equivalent forms that are suitable for the purpose at hand, as the following example shows.

Example	**M.A1HS.42**

Which is the simpler form? A particularly rich mathematical investigation involves finding a general expression for the sum of the first n consecutive natural numbers:

$$S = 1+2+3+\cdots+(n-2)+(n-1)+n.$$

A famous tale speaks of a young C. F. Gauss being able to add the first 100 natural numbers quickly in his head, wowing his classmates and teachers alike. One way to find this sum is to consider the "reverse" of the sum:

$$S = n+(n-1)+(n-2)+\cdots+3+2+1$$

Then the two expressions for S are added together:

$$2S = (n+1)+(n+1)+\cdots+(n+1)+(n+1)+(n+1),$$

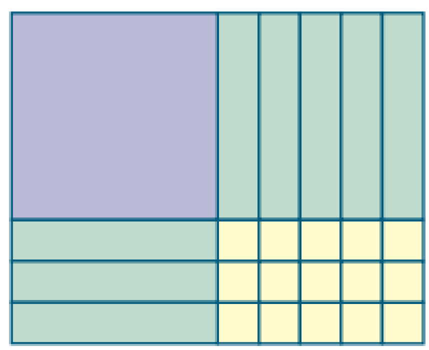
where there are n terms of the form $(n+1)$. Thus, $2S = n(n+1)$, so that $S = \frac{n(n+1)}{2}$. While students may be tempted to transform this expression into $\frac{1}{2}n^2 + \frac{1}{2}n$, they are obscuring the ease with which they can evaluate the first expression. Indeed, since n is a natural number, one of either n or $n+1$ is even, so evaluating $\frac{n(n+1)}{2}$, especially mentally, is often easier. In Gauss's case, $\frac{100(101)}{2} = 50(101) = 5050$.

Students also use different forms of the same expression to reveal important characteristics of the expression. For instance, when working with quadratics, they complete the square in the expression $x^2 - 3x + 4$ to obtain the equivalent expression $\left(x - \frac{3}{2}\right)^2 + \frac{7}{4}$. Students can then reason with the new expression that the term being squared is always greater than or equal to 0; hence, the value of the expression will always be greater than or equal to $\frac{7}{4}$ (**M.A1HS.43, MHM3**). A spreadsheet or a computer algebra system may be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave, further contributing to students' understanding of work with expressions (**MHM5**).

Expressions and Equations
Perform arithmetic operations on polynomials. **M.A1HS.44** Recognize that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. Instructional Note: Focus on polynomial expressions that simplify to forms that are linear or quadratic in a positive integer power of x.

In Algebra I, students begin to explore the set of polynomials in x as a system in its own right, subject to certain operations and properties. To perform operations with polynomials meaningfully, students are encouraged to draw parallels between the set of integers—wherein integers can be added, subtracted, and multiplied according to certain properties—and the set of all polynomials with real coefficients (**M.A1HS.44, MHM7**). If the function concept is developed before or concurrently with the study of polynomials, then a polynomial can be identified with the function it defines. In this way, $x^2 - 2x - 3$, $(x + 1)(1 - 3)$, and $(x - 1)^2 - 4$ are all the same polynomial because they all define the same function.

Algebra Tiles



The rectangle above has height $(x+3)$ and base $(x+5)$. The total area represented, the product of these binomials, is seen to be $x^2 + 5x + 3x + 15 = x^2 + 8x + 15$.

In Algebra I, students are required only to add linear or quadratic polynomials and to multiply linear polynomials to obtain quadratic polynomials, since in later courses they will explore polynomials of higher degree. Students fluently add, subtract, and multiply linear expressions of the form $ax + b$, and add and subtract expressions of the form $ax^2 + bx + c$, with a, b, and c real numbers, understanding that the result is yet another expression of one of these forms. The explicit notion of *closure* of the set of polynomials need not be explored in Algebra I.

Manipulatives such as “algebra tiles” may be used to support understanding of addition and subtraction of polynomials and the multiplication of monomials and binomials. Algebra tiles may be used to offer a concrete representation of the terms in a polynomial (**MHM5**). The tile representation relies on the area interpretation of multiplication: the notion that the product ab can be thought of as the area of a rectangle of dimensions a units and b units. With this understanding, tiles can be used to represent 1 square unit (a 1 by 1 tile), x square units (a 1 by x tile), and x^2 square units (an x by x tile). Finding the product $(x + 5)(x + 3)$ amounts to finding the area of an abstract rectangle of dimensions $(x + 5)$ and $(x + 3)$, as illustrated in the figure (**MHM2**).

Care must be taken in the way negative numbers are handled with this representation. The tile representation of polynomials is also very useful for understanding the notion of completing the square, as described later in this chapter.

Relationships between Quantities and Reasoning with Equations

Create equations that describe numbers or relationships.

M.A1HS.5

Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. Instructional Note: Limit to linear and exponential equations, and, in the case of exponential equations, limit to situations requiring evaluation of exponential functions at integer inputs.

M.A1HS.6

Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. Instructional Note: Limit to linear and exponential equations, and, in the case of exponential equations, limit to situations requiring evaluation of exponential functions at integer inputs.

M.A1HS.7

Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. (e.g., Represent inequalities describing nutritional and cost constraints on combinations of different foods.) Instructional Note: Limit to linear equations and inequalities.

M.A1HS.8
Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. (e.g., Rearrange Ohm's law V = IR to highlight resistance R.) Instructional Note: Limit to formulas with a linear focus.

Expressions and Equations

Create equations that describe numbers or relationships.

M.A1HS.45
Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. Instructional Note: Extend work on linear and exponential equations in the Relationships between Quantities and Reasoning with Equations unit to quadratic equations.

M.A1HS.46
Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. Instructional Note: Extend work on linear and exponential equations in the Relationships between Quantities and Reasoning with Equations unit to quadratic equations.

M.A1HS.47
Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. (e.g., Rearrange Ohm's law V = IR to highlight resistance R. Instructional Note: Extend work on linear and exponential equations in the Relationships between Quantities and Reasoning with Equations unit to quadratic equations. Extend this standard to formulas involving squared variables.

An equation is a statement of equality between two expressions. The values that make the equation true are the solutions to the equation. An identity, in contrast, is true for all values of the variables; rewriting an expression in an equivalent form often creates identities. The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs of numbers which can be plotted in the coordinate plane. In this set of standards, students create equations to solve problems, they correctly graph the equations on coordinate axes, and they interpret solutions in a modeling context. The following example requires students to understand the multiple variables that appear in a given equation and to reason with them.

Example	**M.A1HS.8, M.A1HS.47**
The height h of a ball at time t seconds thrown vertically upward at a speed of v feet/second is given by the equation $h = +vt - 16t^2$. Write an equation whose solution is: a. the time it takes a ball thrown at a speed of 88 ft/sec to rise 20 feet; b. the speed with which the ball must be thrown to rise 20 feet in 2 seconds. **Solution and Comments:** a. We want $h = 20$, and we are told that $v = 88$, so the equation is $20 = 6 + 88t - 16t^2$. b. We want $h = 20$, and we are told that $t = 2$, so the equation is $20 = 6 + 2v - 16 \bullet 2^2$. Although this is a straightforward example, students must be flexible in seeing some of the variables in the equation as constants when others are given values. In addition, the example does not explicitly state that $v = 88$ or $t = 2$, so students must understand the meaning of the variables in order to proceed with the problem (**MHM1**). Adapted from Illustrative Mathematics 2013h.	

To support **M.A1HS.45**, it is recommended that absolute value be addressed. The basic absolute value function has at least two very useful definitions: (1) a descriptive, verbal definition and (2) a formula definition. A common definition of the absolute value of x is:

$$|x| = the\ distance\ from\ the\ number\ x\ to\ 0\ (on\ a\ number\ line).$$

An understanding of the number line easily yields that, for example, $|0| = 0$, $|7| = 7$, and $|-3.9| = 3.9$. However, an equally valid “formula” definition of absolute value reads:

$$|x| = \{\frac{-x, x < 0}{x, x \geq 0}$$

In other words, $|x|$ is simply x whenever x is 0 or positive, but $|x|$ is the opposite of x whenever x is negative. Either definition can be extended to an understanding of the expression $|x - a|$ as the distance between x and a on a number line, an interpretation that has many uses. For a simple application of this idea, suppose a type of bolt is to be mass-produced in a factory with the specification that its width be 5mm with an error no larger than 0.01mm. If w represents the width of a given bolt produced on the production line, then we want w to satisfy the inequality $|w - 5| \leq 0.01$; that is, the difference between the actual width w and the target width should be less than or equal to 0.01 (**MHM4, MHM6).** Students should become comfortable with the basic properties of absolute values (e.g., $|x| + a \neq |x + a|$) and with solving absolute value equations and interpreting the solution.

In higher mathematics courses, intervals on the number line are often denoted by an inequality of the form $|x - a| \leq d$ for a positive number d. For example, $|x - 2| \leq \frac{1}{2}$ represents the closed interval $1\frac{1}{2} \leq x \leq 2\frac{1}{2}$. This can be seen by interpreting $|x - 2| \leq \frac{1}{2}$ as "the distance from x to 2 is less than or equal to $\frac{1}{2}$" and deciding which numbers fit this description.
On the other hand, in the case where $x - 2 < 0$, we would have $|x - 2| = -(x - 2) \leq \frac{1}{2}$, so that $x \geq 1\frac{1}{2}$. In the case 2 where $x - 2 \geq 0$, we have $|x - 2| = x - 2 \leq 0$, which means that $x \leq 2\frac{1}{2}$. Since we are looking for all x that satisfy both inequalities, the interval is $1\frac{1}{2} \leq x \leq 2\frac{1}{2}$. This shows how the formula definition can be used to find this interval.

Relationships between Quantities and Reasoning with Equations

Understand solving equations as a process of reasoning and explain the reasoning.

M.A1HS.9

Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. Instructional Note: Students should focus on and master linear equations and be able to extend and apply their reasoning to other types of equations in future courses. Students will solve exponential equations with logarithms in Algebra II.

Solve equations and inequalities in one variable.

M.A1HS.10

Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. Instructional Note: Extend earlier work with solving linear equations to solving linear inequalities in one variable and to solving literal equations that are linear in the variable being solved for. Include simple exponential equations that rely only on application of the laws of exponents, such as $5^x = 125$ or $2^x = 1/16$.

Expressions and Equations

Solve equations and inequalities in one variable.

M.A1HS.48

Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Instructional Note: Students should learn of the existence of the complex number system, but will not solve quadratics with complex solutions until Algebra II.

An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions. In Algebra I, students solve linear equations and inequalities in one variable, including equations and inequalities with absolute values and equations with coefficients represented by letters (**M.A1HS.10**). In addition, this is the students' first exposure to quadratic equations, and they learn various techniques for solving them and the relationships between those techniques (**M.A1HS.48**). When solving equations, students make use of the symmetric and transitive properties, as well as properties of equality with regard to operations (e.g., "Equals added to equals are equal"). Standard **M.A1HS.9** requires that in any situation, students can solve an equation *and explain the steps* as resulting from previous true equations and using the aforementioned properties (**MHM3**). In this way, the idea of *proof*, while not explicitly named, is given a prominent role in the solving of equations, and the reasoning and justification process is not simply relegated to a future mathematics course.

On Solving Equations: A written sequence of steps is code for a narrative line of reasoning that would use words such as *if*, *then*, *for all*, and *there exists*. In the process of learning to solve equations, students should learn certain "if–then" moves—for example, "If $x = y$, then $x + c = y + c$ for any c." The danger in learning algebra is that students emerge with nothing but the moves, which may make it difficult to detect incorrect or made-up moves later on. Thus the first requirement in this domain is that students understand that solving equations is a process of reasoning (**M.A1HS.9**).

Fragments of Reasoning

$$x^2 = 4$$
$$x^2 - 4 = 0$$
$$(x - 2)(x + 2) = 0$$
$$x = 2, -2$$

This sequence of equations is shorthand for a line of reasoning: *"If x is a number whose* *square is 4, then $x^2 - 4 = 0$, by properties of equality. Since $x^2 - 4 = (x-2)(x+2)$ for all numbers, it follows that $(x-2)(x+2) = 0$. So either $(x-2) = 0$, in which case $x = 2$, or $(x+2) = 0$, in which case $x = -2$."*

Adapted from UA Progressions 2013b, 13.

Students in Algebra I extend their work with exponents to working with quadratic functions and equations that have real roots. To extend their understanding of these quadratic expressions and the functions they define, students investigate properties of quadratics and their graphs in the Functions conceptual category.

Example **M.A1HS.48a**

Standard **M.A1HS.48a** calls for students to solve quadratic equations of the form $(x-p)^2 = q$. In doing so, students rely on the understanding that they can take square roots of both sides of the equation to obtain the following:

$$\sqrt{(x-p)^2} = \sqrt{q} \qquad (1)$$

In the case where $\sqrt{q}$ is a real number, we can solve this equation for x. A common mistake is to introduce the symbol ± here, without understanding its source. Doing so without care often leads students to think that $\sqrt{9} = \pm 3$, for example.

Note that the quantity $\sqrt{a^2}$ is simply a when $a \geq 0$ (as in $\sqrt{5^2} = \sqrt{25} = 5$), while $\sqrt{a^2}$ is equal to $-a$ (the opposite of a) when $a < 0$ (as in $\sqrt{(-4)^2} = \sqrt{16} = 4$). But this means that $\sqrt{a^2} = |a|$. Applying this to equation (1), shown above, yields $|x-p| = \sqrt{q}$. Solving this simple absolute value equation yields that $(x-p) = \sqrt{q}$ or $-(x-p) = \sqrt{q}$. This results in the two solutions $p + \sqrt{p}$, $p - \sqrt{q}$.

Students also transform quadratic equations into the form $ax^2 + bx + c = 0$ for $a \neq 0$, which is the *standard form* of a quadratic equation. In some cases, the quadratic expression factors nicely, and students can apply the zero product property of the real numbers to solve the resulting equation. The *zero product property* states that for two real numbers m and n, $m \bullet n = 0$ if and only if either $m = 0$ or $n = 0$. Hence, when a quadratic polynomial can be rewritten as $a(x-r)(x-s) = 0$, the solutions can be found by setting each of the linear factors equal to 0 separately and obtaining the solution set $\{r, s\}$. In other cases, a means for solving a quadratic equation arises by *completing the square*. Assuming for simplicity that $a = 1$ in the standard equation above, and that the equation has been rewritten as $x^2 + bx = -c$, we can "complete the square" by adding the square of half the coefficient of the x-term to each side of the equation:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2 \qquad (2)$$

The result of this simple step is that the quadratic on the left side of the equation is a perfect square, as shown here:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Thus, we have now converted equation (2) into an equation of the form $(x - p)^2 = q$:

$$\left(x + \frac{b}{x}\right)^2 = -c + \frac{b^2}{4}$$

This equation can be solved by the method described above, as long as the term on the right is non-negative. When $a \neq 1$, the case can be handled similarly and ultimately results in the familiar quadratic formula. Tile representations of quadratics illustrate that the process of completing the square has a geometric interpretation that explains the origin of the name. Students should be encouraged to explore these representations in order to make sense out of the process of completing the square (**MHM1, MHM5**). Completing the square is an example of a recurring theme in algebra: finding ways of transforming equations into certain standard forms that have the same solutions.

Example: Completing the Square **M.A1HS.48a**

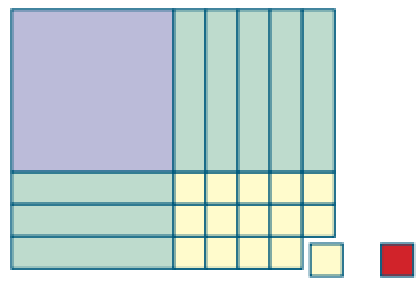
The method of completing the square is a useful skill in algebra. It is generally used to change a quadratic in standard form, $ax^2 + bx + c$, into one in vertex form, $a(x-h)^2 + k$. The vertex form can help determine several properties of quadratic functions. Completing the square also has applications in geometry (**M.GHS.39**) and later higher mathematics courses.

To complete the square for the quadratic $y = x^2 + 8x + 15$, we take half the coefficient of the x-term and square it to yield 16. We realize that we need only to add 1 and subtract 1 to the quadratic expression:

$$
\begin{aligned}
y &= x^2 + 8x + 15 + 1 - 1 \\
&= x^2 + 8x + 16 - 1.
\end{aligned}
$$

Factoring gives us $y = (x+4)^2 - 1$.

In the picture at right, note that the tiles used to represent $x^2 + 8x + 15$ have been rearranged to try to form a square, and that a positive unit tile and a "negative" unit tile are added into the picture to "complete the square."



The same solution techniques used to solve equations can be used to rearrange formulas to highlight specific quantities and explore relationships between the variables involved. For example, the formula for the area of a trapezoid, $A = \left(\frac{b_1+b_2}{2}\right)h$, can be solved for h using the same deductive process (**MHM7, MHM8**). As will be discussed later, functional relationships can often be explored more deeply by rearranging equations that define such relationships; thus, the ability to work with equations that have letters as coefficients is an important skill.

Linear and Exponential Relationships

Solve systems of equations.

M.A1HS.13

Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

M.A1HS.14

Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables. Instructional Note: Build on student experiences graphing and solving systems of linear equations from middle school to focus on justification of the methods used. Include cases where the two equations describe the same line (yielding infinitely many solutions) and cases where two equations describe parallel lines (yielding no solution); connect to standards in Geometry which require students to prove the slope criteria for parallel lines.

Expressions and Equations

Solve systems of equations.

M.A1HS.49

Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$. Instructional Note: Include systems consisting of one linear and one quadratic equation. Include systems that lead to work with fractions. For example, finding the intersections between $x^2 + y^2 = 1$ and $y = (x+1)/2$ leads to the point (3/5, 4/5) on the unit circle, corresponding to the Pythagorean triple $3^2 + 4^2 = 5^2$.

Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system. The process of adding one equation to another is understood in this way: if the two sides of one equation are equal, and the two sides of another equation are equal, then the sum (or difference) of the left sides of the two equations is equal to the sum (or difference) of the right sides. The reversibility of these steps justifies that we achieve an equivalent system of equations by doing this. This crucial point should be consistently noted when reasoning about solving systems of equations (UA Progressions Documents 2013b, 11).

Example **M.A1HS.14**

Solving simple systems of equations. To get started with understanding how to solve systems of equations by linear combinations, students can be encouraged to interpret a system in terms of real-world quantities, at least in some cases. For instance, suppose one wanted to solve this system:

$$3x + y = 40$$
$$4x + 2y = 58$$

Now consider the following scenario: Suppose 3 CDs and a magazine cost $40, while 4 CDs and 2 magazines cost $58.

- What happens to the price when you add 1 CD and 1 magazine to your purchase?

- What is the price if you decided to buy only 2 CDs and no magazine?

Answering these questions amounts to realizing that since $(3x + y) + (x + y) = 40 + 18$, we must have that $x + y = 18$. Therefore, $(3x + y) + (-1)(x + y) = 40 + (-1)18$, which implies that $2x = 22$, or 1 CD costs $11. The value of y can now be found using either of the original equations: $y = 7$.

When solving systems of equations, students also make frequent use of substitution—for example, when solving the system $2x - 9y = 5$ and $y = \frac{1}{3}x + 1$, the expression $\frac{1}{3}x + 1$ can be substituted for y in the first equation to obtain $2x - 9\left(\frac{1}{3}x + 1\right) = 5$. Students also solve such systems approximately, by using graphs and tables of values (**M.A1HS.13, M.A1HS.14, M.A1HS.49**).

Linear and Exponential Relationships

Represent and solve equations and inequalities graphically.

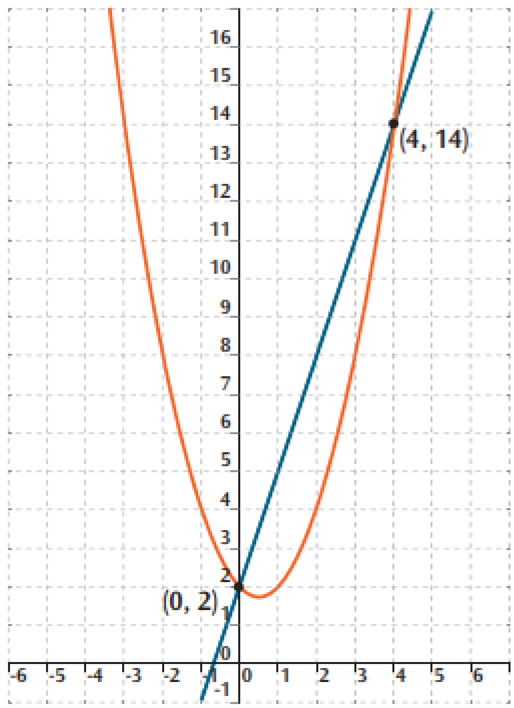
M.A1HS.15
Recognize that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). Instructional Note: Focus on linear and exponential equations and be able to adapt and apply that learning to other types of equations in future courses.

M.A1HS.16
Explain why the x-coordinates of the points where the graphs of the equations y = f(x) and y = g(x) intersect are the solutions of the equation f(x) = g(x); find the solutions approximately (e.g., using technology to graph the functions, make tables of values or find successive approximations). Include cases where f(x) and/or g(x) are linear, polynomial, rational, absolute value, exponential and logarithmic functions. Instructional Note: Focus on cases where f(x) and g(x) are linear or exponential.

M.A1HS.17
Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

One of the most important goals of instruction in mathematics is to illuminate connections between different mathematical concepts. In particular, in standards **M.A1HS.15– M.A1HS.17**, students learn the relationship between the algebraic representation of an equation and its graph plotted in the coordinate plane and understand geometric interpretations of solutions to equations and inequalities. In Algebra I, students work only with linear, exponential, quadratic, step, piecewise, and absolute value functions. As students become more comfortable with function notation — for example, writing $f(x) = 3x - 2$ and $g(x) = x^2 - x + 2$— they begin to see solving the equation $3x - 2 = x^2 - x + 2$ as solving the equation $f(x) = g(x)$.That is, they find those x-values where two functions take on the same output value. Moreover, they graph the two equations (see the following figure) and see that the x-coordinate(s) of the point(s) of intersection of the graphs of $y = f(x)$ and $y = g(x)$ are the solutions to the original equation.

Graphs of $y = 3x + 2$ **and** $y = x^2 - x + 2$



Students also create tables of values for functions to approximate or find exact solutions to equations, such as those plotted in the figure. For example, they may use spreadsheet software to construct a table, such as the one shown.

Values for $f(x) = 3x + 2$ **and** $g(x) = x^2 - x + 2$

x	$f(x) = 3x + 2$	$g(x) = x^2 - x + 2$
-6	-16	44
-5	-13	32
-4	-10	22
-3	-7	14
-2	-4	8
-1	-1	4
0	2	2
1	5	2
2	8	4
3	11	8
4	14	14
5	17	22
6	20	32
7	23	44
8	26	58
9	29	74

Although a table like this one does not offer sufficient proof that all solutions to a given equation have been found, students can reason in certain situations why they have found all solutions (**MHM3, MHM6**). In this example, since the original equation is of degree two, we know that there are at most two solutions, so that the solution set is {0, 4}.

Conceptual Category: Statistics and Probability

In Algebra I, students build on their understanding of key ideas for describing distributions—shape, center, and spread—presented in the standards for grades six through eight. This enhanced understanding allows them to give more precise answers to deeper questions, often involving comparisons of data sets.

Descriptive Statistics
Summarize, represent, and interpret data on a single count or measurement variable.
M.A1HS.33 Represent data with plots on the real number line (dot plots, histograms, and box plots).
M.A1HS.34 Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets. Instructional Note: In grades 6 – 8, students describe center and spread in a data distribution. Here they choose a summary statistic appropriate to the characteristics of the data distribution, such as the shape of the distribution or the existence of extreme data points.

M.A1HS.35
Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers). Instructional Note: In grades 6 – 8, students describe center and spread in a data distribution. Here they choose a summary statistic appropriate to the characteristics of the data distribution, such as the shape of the distribution or the existence of extreme data points.

Summarize, represent, and interpret data on two categorical and quantitative variables.

M.A1HS.36
Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal and conditional relative frequencies). Recognize possible associations and trends in the data.

M.A1HS.37
Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.

a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear and exponential models.
b. Informally assess the fit of a function by plotting and analyzing residuals. Instructional Note: Focus should be on situations for which linear models are appropriate.
c. Fit a linear function for scatter plots that suggest a linear association.

Instructional Note: Students take a more sophisticated look at using a linear function to model the relationship between two numerical variables. In addition to fitting a line to data, students assess how well the model fits by analyzing residuals.

Interpret linear models.

M.A1HS.38
Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. Instructional Note: Build on students' work with linear relationships in eighth grade and introduce the correlation coefficient. The focus here is on the computation and interpretation of the correlation coefficient as a measure of how well the data fit the relationship.

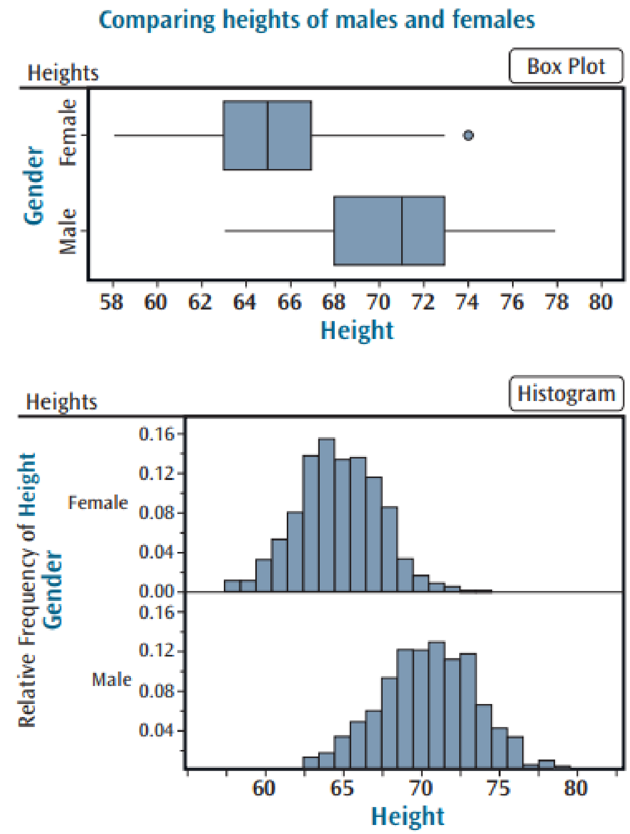
M.A1HS.39
Compute (using technology) and interpret the correlation coefficient of a linear fit. Instructional Note: Build on students' work with linear relationships in eighth grade and introduce the correlation coefficient. The focus here is on the computation and

interpretation of the correlation coefficient as a measure of how well the data fit the relationship.

M.A1HS.40
Distinguish between correlation and causation. Instructional Note: The important distinction between a statistical relationship and a cause-and-effect relationship is the focus.

Standards **M.A1HS.33– M.A1HS.37** extend concepts that students begin learning in grades six through eight and, as such, may be considered supporting standards for **M.A1HS.38– M.A1HS.40**. In general, students use shape and the question(s) to be answered to decide on the median or mean as the more appropriate measure of center and to justify their choice through statistical reasoning. Students may use parallel box plots or histograms to compare differences in the shape, center, and spread of comparable data sets (**M.A1HS.33– M.A1HS.34**).

Example	**M.A1HS.34**
The following graphs show two ways of comparing height data for males and females in the 20–29 age group. Both involve plotting the data or data summaries (box plots or histograms) on the same scale, resulting in what are called parallel (or side-by-side) box plots and parallel histograms (**M.A1HS.33**). The parallel box plots show an obvious difference in the medians and the interquartile ranges (IQRs) for the two groups; the medians for males and females are 71 inches and 65 inches, respectively, while the IQRs are 5 inches and 4 inches. Thus, male heights center at a higher value but are slightly more variable. The parallel histograms show the distributions of heights to be mound-shaped and fairly symmetrical (approximately normal) in shape. Therefore, the data can be succinctly described using the mean and standard deviation. Heights for males and females have means of 70.4 inches and 64.7 inches, respectively, and standard deviations of 3.0 inches and 2.6 inches. Students should be able to sketch each distribution and answer questions about it solely from knowledge of these three facts (shape, center, and spread). For either group, about 68% of the data values will be within one standard deviation of the mean (**M.A1HS.34–35**). Students also observe that the two measures of center—median and mean—tend to be close to each other for symmetric distributions.	



Heights of U.S. males and females in the 20–29 age group

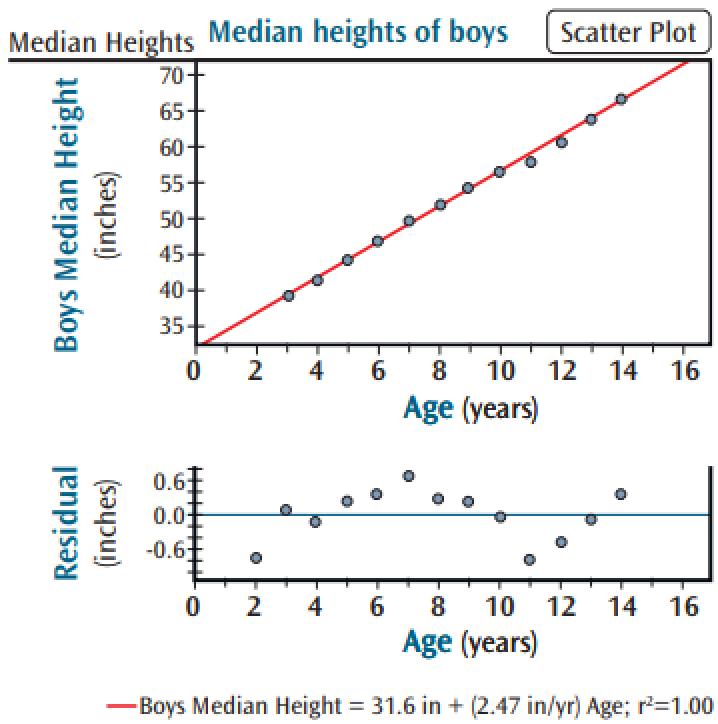
Source: United States Census Bureau 2009 (Statistical Abstract of the United States, Table 201).

Adapted from UA Progressions Documents 2012d, 3.

As with univariate data analysis, students now take a deeper look at bivariate data, using their knowledge of proportions to describe categorical associations and using their knowledge of functions to fit models to quantitative data (**M.A1HS.36– M.A1HS.37**). Students have seen scatter plots in the grade-eight standards and now extend that knowledge to fit mathematical models that capture key elements of the relationship between two variables and to explain what the model tells us about the relationship. Students must learn to examine scatter plots carefully, as the “obvious” pattern does not always tell the whole story and may prove misleading. A line of best fit may appear to fit data almost perfectly, while an examination of the *residuals*—the collection of differences between corresponding coordinates on a least squares line and the actual data value for a variable—may reveal more about the behavior of the data.

Example	**M.A1HS.37b**

The graphs below show the median heights of growing boys from the ages of 2 through 14. The line (least squares regression line) with slope 2.47 inches per year of growth looks to be a perfect fit (**M.A1HS.37c**). However, the residuals—the differences between the corresponding coordinates on the least squares line and the actual data values for each age—reveal additional information. A plot of the residuals shows that growth does not proceed at a constant rate over those years.



Source: Centers for Disease Control and Prevention (CDC) 2002.

Adapted from UA Progressions Documents 2012d, 5.

Finally, students extend their work from topics covered in the grade-eight standards and other topics in Algebra I to interpret the parameters of a linear model in the context of data that it represents. They compute *correlation coefficients* using technology and interpret the value of the coefficient (**MHM4, MHM5**). Students see situations where correlation and causation are mistakenly interchanged, and they carefully examine the story that data and computed statistics are trying to tell (**M.A1HS.38– M.A1HS.40**).

Common Misconceptions – By Domain

Number and Quantity

- Students often have difficulty differentiating between finding the solution to an equation such as $x^2 = 25$ and taking the square root of a number. The equation $x^2 = 25$ has two solutions, 5 and -5. Expressions such as $\sqrt{3^2}$ or $\sqrt{(-7)^2}$ ask for the "principal" square root. As a result, $\sqrt{3^2}$ = 3 and $\sqrt{(-7)^2}$ = 7

- Student confusion in differentiating between expressions involving the square of a negative number and the negative of a square, often extend to square roots. Students may need help in appreciating the Order of Operations in differentiating between the expressions.

$$(-5)^2 \rightarrow (-5)(-5) \rightarrow 25$$

$$-(5)^2 \rightarrow -(5)(5) \rightarrow -25$$

$$\sqrt{(-5)^2} \rightarrow \sqrt{25} \rightarrow 5$$

$$-\sqrt{5^2} \rightarrow -\sqrt{25} \rightarrow -5$$

$$\sqrt{(5)^2} \rightarrow \sqrt{-25} \rightarrow 5$$

- Students may have difficulty when negative exponents involve fractions. Students may incorrectly believe that $16^{-\frac{1}{2}}$ means -4 rather than $\frac{1}{4}$.

$$16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$$

Algebra

- Students may have difficulty distinguishing between the terms of an expression and the variable of an expression. For example, students may reason that the trinomial $3a + b + 5cd$ has four terms because the student is counting variable rather than terms. Often students do not recognize a constant as a term and incorrectly identify the trinomial $4x - 2y + 7$ as having two terms.

- Students who rely solely on procedures may believe they need multiply the binomials in an equation such as $(x + 4)(x - 3) = 0$ and then factor the resulting expression to solve for the zeros. Students need to develop an understanding of the concept of the Zero Factor Property. Students may experience similar confusion in applying the Zero Factor Property in solving equations such as $5(x-7)(x+1)=0$ or $6x(x-7)=0$.

- Students who mistakenly believe that $\sqrt{x^2+\ y^2}$ is equivalent to $x+y$, may deduce this from the misconception that $(x+y)^2$ is equivalent to $x^2+\ y^2$.

- Students who mistakenly equate $(x+y)^2$ with $x^2+\ y^2$ or equate $(x-y)^2$ with x^2-y^2 have misconceptions regarding the concept of finding the square of a number or expression. These misconceptions may stem from students' prior difficulty in recognizing that $(-5)^2$ → (-5)(-5) →25 or that $-(5)^2$ → –(5)(5) → -25.

- Students who incorrectly equate rational expressions such as $\frac{x^2-6x+9}{x-3}$ with $x-2x+9$ or $x-6x-3$ or x^2+2x+9, etc., have difficulty distinguishing between terms and factors.

- Students may demonstrate difficulties in solving the equation $\sqrt{x-1}$ = $x-7$. Students may appropriately decide to square both sides of the equation and write $(\sqrt{x-1}\,)^2$ = $(x-7)^2$ Student error may arise in writing $(\sqrt{x-1}\,)^2$ = x^2 - 7^2, demonstrating misconceptions in squaring binomials (or multiplying) binomials.

- The procedure for solving equations and inequalities are so similar. As a result, students often forget to attend to precision when multiplying and dividing by a negative number when solving inequalities. Students may correctly solve equations such as $-2x=6$ and find its solution to be $x=-3$. Misconceptions arise when students equate solving the equation with a similar inequality $-2x\ \leq 6$ and incorrectly determine that $x\ \leq -3$. The misconception can be addressed by encouraging students to verify solutions to equations and to inequalities.

- Students often solve rational and radical equations without checking to determine if any solution may be erroneous. Students should be encouraged to verify solutions.

Functions

- Students may erroneously interpret the notation $g(3)$ to mean "$g\ times\ 3$".

- Students often believe that all functions must use the symbols f, x, and y.

- When graphing students may confuse the parts of the slope-equation form of a linear equation. Students may incorrectly determine that the function $g(x) = 2x + \frac{3}{5}$ has a slope of $\frac{3}{5}$ and a y-intercept of 2. Students may be relying on a procedural understanding that the y-intercept is always an integer and the slope or $\frac{rise}{run}$ must be a fraction.

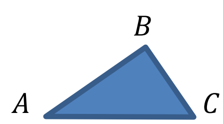
- Students often incorrectly assume that the function $f(x + k)$ where $k > 0$ will result in a horizontal shift of the graph k units to the right.

Geometry

- Students often think of congruence as "figures with the same shape and size." While this understanding is not incorrect, it is important to that the continually emphasize the link of congruence with rigid motions and show that rigid motions do in fact produce "figures with the same shape and size."

- Students may look at scale factor as the distance that is added on to the original distance. Dilations where the center of dilation is a vertex of a figure can prove challenging because the sides of the pre-image and image overlap.

- Students may have difficulty identifying the relevant sides in relationship to a given angle, especially If the triangle is not depicted in a typical position where its legs are horizontal and vertical. For example, given ΔABC below with right angle B, students may struggle to understand that $\sin A = \frac{BC}{AC}$ and that $\sin A \neq \frac{BC}{AB}$.



- Students may have difficulty differentiating between the meaning of the statements $\Delta ABC \cong \Delta DEF$ and $\Delta ABC \sim \Delta DEF$. While $\Delta ABC \cong \Delta DEF$ implies that AB = DE, $\Delta ABC \sim \Delta DEF$ does not imply that AB = DE.

- Students often have difficulty differentiating among $\overleftarrow{EF}$, $\overrightarrow{EF}$, $\overleftrightarrow{EF}$, and $\overline{EF}$.

Statistics and Probability

- Students often use the word *outlier* inaccurately, failing to verify that it satisfies the necessary conditions. Using terms such as *unusual feature* or *data point* can help students avoid using the term outlier when it is not appropriate.

- Students commonly believe that any data that is collected should follow a normal distribution.

- Students need to understand that association does not necessarily provide evidence for cause and effect.

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