West Virginia math4life

Educators' Guide for Mathematics

Algebra II

West Virginia DEPARTMENT OF EDUCATION

Algebra II

The Algebra II course extends students' understanding of functions and real numbers and increases the tools students have for modeling the real world. Students in Algebra II extend their notion of *number* to include complex numbers and see how the introduction of this set of numbers yields the solutions of polynomial equations and the Fundamental Theorem of Algebra. Students deepen their understanding of the concept of *function* and apply equation-solving and function concepts to many different types of functions. The system of polynomial functions, analogous to integers, is extended to the field of rational functions, which is analogous to rational numbers. Students explore the relationship between exponential functions and their inverses, the logarithmic functions. Trigonometric functions are extended to all real numbers, and their graphs and properties are studied. Finally, students' knowledge of statistics is extended to include understanding *the normal distribution*, and students are challenged to make inferences based on sampling, experiments, and observational studies.

For the Traditional Pathway, the standards in the Algebra II course come from the following conceptual categories: modeling, functions, number and quantity, algebra, and statistics and probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not simply topics to be checked off from a list during isolated units of instruction; rather, they represent content that should be present throughout the school year in rich instructional experiences.

What Students Learn in Algebra II

Building on their work with linear, quadratic, and exponential functions, students in Algebra II extend their repertoire of functions to include polynomial, rational, and radical functions. (In Algebra II, rational functions are limited to those with numerators having a degree not more than 1 and denominators having a degree not more than 2; radical functions are limited to square roots or cube roots of at most quadratic polynomials.) Students work closely with the expressions that define the functions and continue to expand and hone their abilities to model situations and to solve equations, including solving quadratic equations over the set of complex numbers and solving exponential equations using the properties of logarithms. Based on their previous work with functions, and on their work with trigonometric ratios and circles in Geometry, students now use the coordinate plane to extend trigonometry to model periodic phenomena. They explore the effects of transformations on graphs of diverse functions, including functions arising in applications, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of underlying function. They identify appropriate types of functions to model a situation, adjust parameters to improve the model, and compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. Students see how the visual displays and summary statistics learned in earlier grade levels relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and the role of randomness and careful design in the conclusions that can be drawn.

Examples of Key Advance from Previous Grade Levels or Courses

- In Algebra I, students added, subtracted, and multiplied polynomials. Students in Algebra II divide polynomials that result in remainders, leading to the factor and remainder theorems. This is the underpinning for much of advanced algebra, including the algebra of rational expressions.
- Themes from middle-school algebra continue and deepen during high school. As early as grade six, students begin thinking about solving equations as a process of reasoning (**M.6.16**). This perspective continues throughout Algebra I and Algebra II. "Reasoned solving" plays a role in Algebra II because the equations students encounter may have extraneous solutions (**M.A1HS.16**).
- In Algebra I, students work with quadratic equations with no real roots. In Algebra II, they extend their knowledge of the number system to include complex numbers, and one effect is that they now have a complete theory of quadratic equations: Every quadratic equation with complex coefficients has (counting multiplicity) two roots in the complex numbers.
- In grade eight, students learn the Pythagorean Theorem and used it to determine distances in a coordinate system (**M.8.21–M.8.23**). In the Geometry course, students prove theorems using coordinates (**M.GHS.29–M.GHS.32**). In Algebra II, students build on their understanding of distance in coordinate systems and draw on their growing command of algebra to connect equations and graphs of conic sections (for example, refer to standard **M.GHS.39**).
- In Geometry, students begin trigonometry through a study of right triangles. In Algebra II, they extend the three basic functions to the entire unit circle.
- As students acquire mathematical tools from their study of algebra and functions, they apply these tools in statistical contexts (for example, refer to standard **M.A1HS.37**). In a modeling context, students might informally fit an exponential function to a set of data, graphing the data and the model function on the same coordinate axes.

Connecting Mathematical Habits of Mind and Content

The Mathematical Habits of Mind (MHM) apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, relevant, and meaningful subject. The Mathematical Habits of Mind represent a picture of what it looks like for students to *do mathematics* and, to the extent possible, content instruction should include attention to appropriate practice standards. There are ample opportunities for students to engage in each mathematical practice in Algebra II; the table below offers some general examples.

Mathematical Habits of Mind—Explanation and Examples for Algebra II

Mathematical Habits of Mind	Explanation and Examples
MHM1 Make sense of problems and persevere in solving them.	Students apply their understanding of various functions to real-world problems. They approach complex mathematics problems and break them down into smaller problems, synthesizing the results when presenting solutions.
MHM2 Reason abstractly and quantitatively.	Students deepen their understanding of variables—for example, by understanding that changing the values of the parameters in the expression $A\sin(Bx+C)+D$ has consequences for the graph of the function. They interpret these parameters in a real-world context.
MHM3 Construct viable arguments and critique the reasoning of others.	Students continue to reason through the solution of an equation and justify their reasoning to their peers. Students defend their choice of a function when modeling a real-world situation.
MHM4 Model with mathematics.	Students apply their new mathematical understanding to real-world problems, making use of their expanding repertoire of functions in modeling. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
MHM5 Use appropriate tools strategically.	Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions.
MHM6 Attend to precision.	Students make note of the precise definition of complex number, understanding that real numbers are a subset of complex numbers. They pay attention to units in real-world problems and use unit analysis as a method for verifying their answers.
MHM7 Look for and make use of structure.	Students see the operations of *complex numbers* as extensions of the operations for real numbers. They understand the periodicity of sine and cosine and use these functions to model periodic phenomena.
MHM8 Look for and express regularity in repeated reasoning.	Students observe patterns in geometric sums—for example, that the first several sums of the form $\sum_{k=0}^{n} 2^k$ can be written as follows: $1 = 2^1 - 1$ $1+2 = 2^2 - 1$

	$$1+2+4 = 2^3 - 1$$ $$1+2+4+8 = 2^4 - 1$$ Students use this observation to make a conjecture about any such sum.

MHM4 holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. In the description of the Algebra II content standards that follow, Modeling is covered first to emphasize its importance in the higher mathematics curriculum. Examples where specific Mathematical Habits of Mind can be implemented in the Algebra II standards are noted in parentheses, with the standard(s) also listed.

Algebra II Content Standards

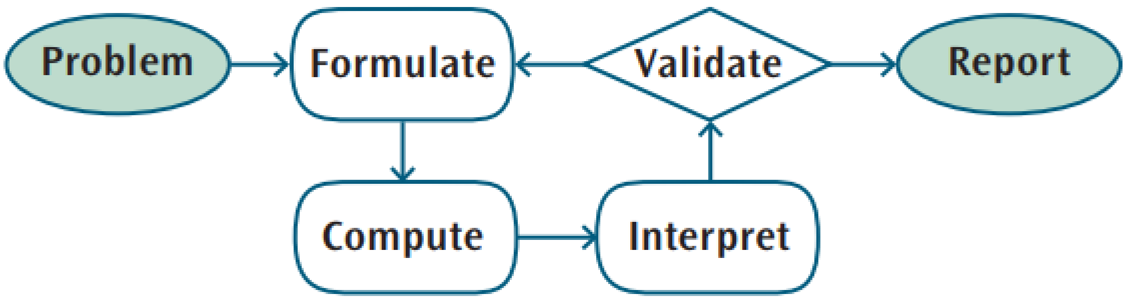
The Algebra II course is organized by domains, clusters, and standards. The overall purpose and progression of the standards included in Algebra II are described below, according to each conceptual category. Standards that are considered new for teachers of secondary-grades are discussed more thoroughly than other standards.

Conceptual Category: Modeling

Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them, and the mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise: Which of the quantities present in this situation are known and unknown? Can a table of data be made? Is there a functional relationship in this situation? Students need to decide on a solution path, which may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They use previously derived models (e.g., linear functions), but may find that a new equation or function will apply. In addition, students may discover that answering their question requires solving an equation that involves knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a mathematical model (an equation, table, graph, or the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see the diagram below of a modeling cycle. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Modeling Cycle



The examples for this course are framed as much as possible to illustrate the concept of mathematical modeling. The important ideas surrounding polynomial and rational functions, graphing, trigonometric functions and their inverses, and applications of statistics are explored through this lens.

Conceptual Category: Functions

Work on functions begins in Algebra I. In Algebra II, students encounter more sophisticated functions, such as polynomial functions of degree greater than 2, exponential functions having all real numbers as the domain, logarithmic functions, and extended trigonometric functions and their inverses. Several standards of the functions category are repeated here, illustrating that the standards attempt to reach depth of understanding of the *concept* of a function. As stated in the University of Arizona (UA) Progressions Documents, “students should develop ways of thinking that are general and allow them to approach any function, work with it, and understand how it behaves, rather than see each function as a completely different animal in the bestiary” (UA Progressions Documents 2013c, 7). For instance, students in Algebra II see quadratic, polynomial, and rational functions as belonging to the same system.

Polynomial, Rational, and Radical Relationships
Analyze functions using different representations.
M.A2HS.18

Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. Instructional Note: Relate this standard to the relationship between zeros of quadratic functions and their factored forms.

Modeling with Functions

Interpret functions that arise in applications in terms of a context.

M.A2HS.27

For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. Instructional Note: Emphasize the selection of a model function based on behavior of data and context.

M.A2HS.28

Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. (e.g., If the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.) Note: Emphasize the selection of a model function based on behavior of data and context.

M.A2HS.29

Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. Note: Emphasize the selection of a model function based on behavior of data and context.

Analyze functions using different representations.

M.A2HS.30

Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.
b. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Instructional Note: Focus on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate.

M.A2HS.31
Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. Instructional Note: Focus on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate.

M.A2HS.32
Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). (e.g., Given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.) Instructional Note: Focus on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate.

In this domain, students work with functions that model data and choose an appropriate model function by considering the context that produced the data. Students' ability to recognize rates of change, growth and decay, end behavior, roots, and other characteristics of functions is becoming more sophisticated; they use this expanding repertoire of families of functions to inform their choices for models. This group of standards focuses on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate (**M.A2HS.27–M.A2HS.32**). The following example illustrates some of these standards.

Example: The Juice Can **M.A2HS.27–M.A2HS.28; M.A2HS.30–M.2HS.32**

Students are asked to find the minimal surface area of a cylindrical can of a fixed volume. The surface area is represented in units of square centimeters (cm^2), the radius in units of centimeters (cm), and the volume is fixed at 355 milliliters (ml), or 355 cm^3. Students can find the surface area of this can as a function of the radius:

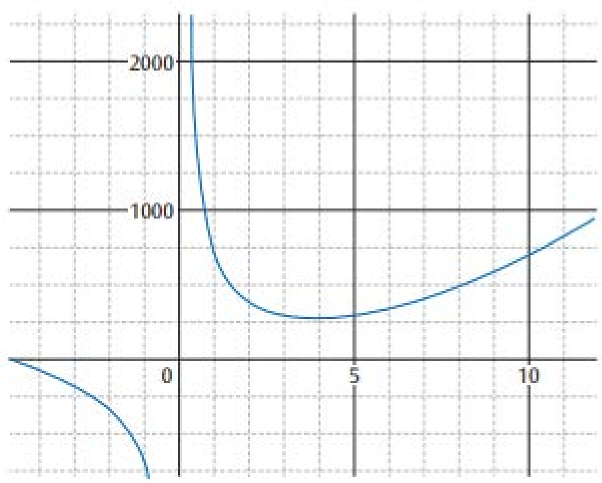
$$S(r) = \frac{2(355)}{r} + 2\pi r^2$$

(See The Juice-Can Equation example that appears in the algebra conceptual category of this chapter.) This representation allows students to examine several things. First, a table of values will provide a hint about the minimal surface area. The table below lists several values for based S on r:

r(cm)	S(cm^2)
0.5	1421.6
1.0	716.3

1.5	487.5
2.0	380.1
2.5	323.
3.0	293.2
3.5	279.8
4.0	278.0
4.5	284.9
5.0	29..0
5.5	319.1
6.0	344.4
6.5	374.6
7.0	409.1
7.5	447.9
8.0	490.7

The data suggest that the minimal surface area occurs when the radius of the base of the juice can is between 3.5 and 4.5 centimeters. Successive approximations using values of between these values will yield a better estimate. But how can students be sure that the minimum is truly located here? A graph $S(r)$ of provides a clue:



Furthermore, students can deduce that as the approximations gets smaller, the term $\frac{2(355)}{r}$ gets larger and larger, while the term $2\pi r$ gets smaller and smaller, and that the reverse is true as the approximations grows larger, so that there is truly a minimum somewhere in the interval $[3.5, 4.5]$.

Graphs help students reason about rates of change of functions (**M.A2HS.29**). In grade eight, students learned that the *rate of change* of a linear function is equal to the slope of the graph of that function. And because the slope of a line is constant, the phrase “rate of change” is clear for linear functions. For non-linear functions, however, rates of change are not constant, and thus average rates of change over an interval are used. For example, for the function defined for all real numbers by $g(x) = x^2$, the average rate of change from $x = 2$ to $x = 5$ is

$$\frac{g(5) - g(2)}{5 - 2} = \frac{25 - 4}{5 - 2} = \frac{21}{3} = 7.$$

This is the slope of the line containing the points $(2, 4)$ and $(5, 25)$ on the graph of g. If g is interpreted as returning the area of a square of side length x, then this calculation means that over this interval the area changes, on average, by 7 square units for each unit increase in the side length of the square (UA Progressions Documents 2013c, 9). Students could investigate similar rates of change over intervals for the Juice Can problem shown previously.

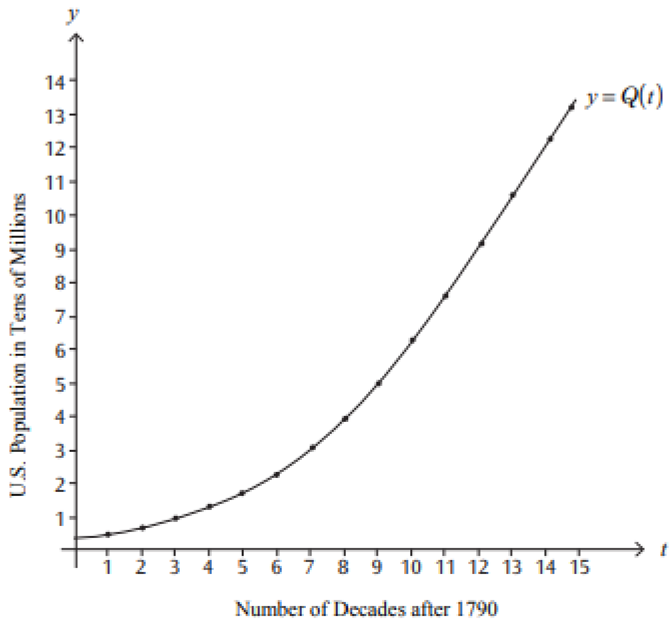
Modeling with Functions
Build a function that models a relationship between two quantities. **M.A2HS.33** Write a function that describes a relationship between two quantities. Combine standard function types using arithmetic operations. (e.g., Build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.) Instructional Note: Develop models for more complex or sophisticated situations than in previous courses. **Build new functions from existing functions.** **M.A2HS.34** Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. Instructional Note: Use transformations of functions to find models as students consider increasingly more complex situations. Observe the effect of multiple transformations on a single graph and the common effect of each transformation across function types. **M.A2HS.35** Find inverse functions. Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. (e.g., f(x) = 2 x3 or f(x) = (x+1)/(x-1)

for x ≠ 1.) Instructional Note: Use transformations of functions to find models as students consider increasingly more complex situations. Extend this standard to simple rational, simple radical, and simple exponential functions; connect this standard to M.A2HS.34.

Students in Algebra II develop models for more complex situations than in previous courses, due to the expansion of the types of functions available to them (**M.A2HS.33**). Modeling contexts provide a natural place for students to start building functions with simpler functions as components. Situations in which cooling or heating are considered involve functions that approach a limiting value according to a decaying exponential function. Thus, if the ambient room temperature is 70 degrees Fahrenheit and a cup of tea is made with boiling water at a temperature of 212 degrees Fahrenheit, a student can express the function describing the temperature as a function of time by using the constant function $f(t) = 70$ to represent the ambient room temperature and the exponentially decaying function $g(t) = 142e^{-kt}$ to represent the decaying difference between the temperature of the tea and the temperature of the room, which leads to a function of this form:$T(t) = 70 + 142e^{-kt}$

Students might determine the constant experimentally (**MHM4, MHM5**).

Example: Population Growth	**M.A2HS.33**
The approximate population of the United States, measured each decade starting in 1790 through 1940, can be modeled with the following function: $$P(t) = \frac{(3{,}900{,}000 \times 200{,}000{,}000)e^{0.31t}}{200{,}000{,}000 + 3{,}900{,}000(e^{0.31t} - 1)}$$ In this function, t represents the number of decades after 1790. Such models are important for planning infrastructure and the expansion of urban areas, and historically accurate long-term models have been difficult to derive.	



Questions:

a. According to this model, what was the population of the United States in the year 1790?

b. According to this model, when did the U.S. population first reach 100,000,000? Explain your answer.

c. According to this model, what should the U.S. population be in the year 2010? Find the actual U.S. population in 2010 and compare with your result.

d. For larger values of t, such as $t = 50$, what does this model predict for the U.S. population? Explain your findings.

Solutions:

a. The population in 1790 is given by $P(0)$, which is easily found to be 3,900,000 because $e^{0.31(0)} = 1$.

b. This question asks students to find such that $P(t) = 100{,}000{,}000$. Dividing the numerator and denominator on the left by 100,000,000 and dividing both sides of the equation by 100,000,000 simplifies this equation to

$$\frac{3.9 \times 2 \times e^{0.31t}}{200 + 3.9(e^{0.31t} - 1)} = 1.$$

Algebraic manipulation and solving for t result in $t \approx \frac{1}{0.31} \ln 50.28 \approx 12.64$. This means that after 1790, it would take approximately 126.4 years for the population to reach 100 million.

c. Twenty-two (22) decades after 1790, the population would be approximately 190,000,000, which is far less (by about 119,000,000) than the estimated U.S. population of 309,000,000

in 2010.
d. The structure of the expression reveals that for very large values of t, the denominator is dominated by $3{,}900{,}000e^{0.31t}$. Thus, for very large values of t,

$$P(t) \approx \frac{3{,}900{,}000 \times 200{,}000{,}000 \times e^{0.31t}}{3{,}900{,}000e^{0.31t}} = 200{,}000{,}000.$$

Therefore, the model predicts a population that stabilizes at 200,000,000 as t increases.

Adapted from Illustrative Mathematics 2013m.

Students can make good use of graphing software to investigate the effects of replacing a function $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for different types of functions (**MHM5**). For example, starting with the simple quadratic function, students see the relationship between these transformed functions and the vertex form of a general quadratic, $f(x) = a(x - h)^2 + k$. They understand the notion of a *family of functions* and characterize such function families based on their properties. These ideas are explored further with trigonometric functions (**M.A2HS.21**).

With standard **M.A2HS.35**, students learn that some functions have the property that an input can be recovered from a given output; for example, the equation $f(x) = c$ can be solved for x, given that c lies in the range of f. Students understand that this is an attempt to "undo" the function, or to "go backwards." Tables and graphs should be used to support student understanding here. This standard dovetails nicely with standard **M.A2HS.36** described below and should be taught in progression with it. Students will work more formally with inverse functions in advanced mathematics courses, and so standard **M.A2HS.36** should be treated carefully to prepare students for deeper understanding of functions and their inverses.

Modeling with Functions

Construct and compare linear, quadratic, and exponential models and solve problems.

M.A2HS.36

For exponential models, express as a logarithm the solution to a b^{ct} = d where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.
Instructional Note: Consider extending this unit to include the relationship between properties of logarithms and properties of exponents, such as the connection between the properties of exponents and the basic logarithm property that log xy = log x +log y.

Students work with exponential models in Algebra I and continue this work in Algebra II. Since the exponential function $f(x) = b^x$ is always increasing or always decreasing for $b \neq 0$, 1, it can be deduced that this function has an inverse, called the *logarithm to the base b*, denoted by $g(x) = \log_b x$. The logarithm has the property that $\log_b x = y$ if and only if $b^y = x$, and this arises in contexts where one wishes to solve an exponential equation. Students find logarithms with base b equal to 2, 10, or e by hand and using technology (**MHM5**). **M.A2HS.36** calls for students to explore the properties of logarithms, such as $\log_b xy = \log_b x + \log_b y$, and students connect these properties to those of exponents (e.g., the previous property comes from the fact that the logarithm represents an exponent and that $b^{n+m} = b^n \bullet b^m$). Students solve problems involving exponential functions and logarithms and express their answers using logarithm notation (**M.A2HS.36**). In general, students understand logarithms as functions that undo their corresponding exponential functions; instruction should emphasize this relationship.

Trigonometric Functions

Extend the domain of trigonometric functions using the unit circle.

M.A2HS.19
Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

M.A2HS.20
Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Model periodic phenomena with trigonometric functions.

M.A2HS.21
Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

Prove and apply trigonometric identities.

M.A2HS.22
Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find sin (θ), cos (θ), or tan (θ), given sin (θ), cos (θ), or tan (θ), and the quadrant of the angle. Instructional Note: An Algebra II course with an additional focus on trigonometry could include the standard "Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems." This could be limited to acute angles in Algebra II.

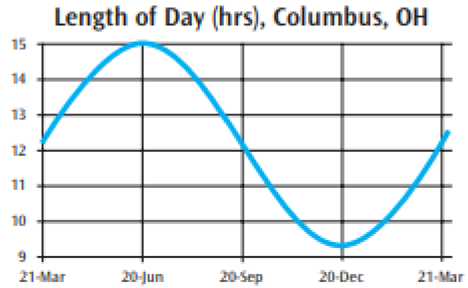
This set of standards calls for students to expand their understanding of the trigonometric functions first developed in Geometry. At first, the trigonometric functions apply only to angles in right triangles; $\sin\theta$, $\cos\theta$, and $\tan\theta$ make sense only for $0 < \theta < \frac{\pi}{2}$. By representing right triangles with hypotenuse 1 in the first quadrant of the plane, it can be seen that $(\cos\theta, \sin\theta)$

represents a point on the unit circle. This leads to a natural way to extend these functions to any value θ of that remains consistent with the values for acute angles: interpreting θ as the radian measure of an angle traversed from the point (1,0) counterclockwise around the unit circle, $\cos\theta$ is taken to be the x-coordinate of the point corresponding to this rotation and $\sin\theta$ to be the y-coordinate of this point. This interpretation of sine and cosine immediately yields the Pythagorean Identity: that $\cos^2\theta + \sin^2\theta = 1$. This basic identity yields others through algebraic manipulation and allows values of other trigonometric functions to be found for a given θ if one of the values is known (**M.A2HS.19– M.A2HS.20, M.A2HS.22**).

The graphs of the trigonometric functions should be explored with attention to the connection between the unit-circle representation of the trigonometric functions and their properties—for example, to illustrate the periodicity of the functions, the relationship between the maximums and minimums of the sine and cosine graphs, zeros, and so forth. Standard **M.A2HS.21** calls for students to use trigonometric functions to model periodic phenomena. This is connected to standard **M.A2HS.32** (families of functions), and students begin to understand the relationship between the parameters appearing in the general cosine function $f(x) = A \bullet \cos(Bx - C) + D$ (and sine function) and the graph and behavior of the function (e.g., amplitude, frequency, line of symmetry).

Example: Modeling Daylight Hours **M.A2HS.21**
By looking at data for length of days in Columbus, Ohio, students see that the number of daylight hours is approximately sinusoidal, varying from about 9 hours, 20 minutes on December 21 to about 15 hours on June 21. The average of the maximum and minimum gives the value for the midline, and the amplitude is half the difference of the maximum and minimum. Approximations of these values are set as $A = 12.17$ and $B = 2.83$. With some support, students determine that for the period to be 365 days (per cycle), or for the frequency to be $\frac{1}{365}$ cycles per day, $C = \frac{2\pi}{365}$, and if day 0 corresponds to March 21, no phase shift would be needed, so $D = 0$. Thus, $f(t) = 12.17 + 2.83\sin\left(\frac{2\pi t}{365}\right)$ is a function that gives the approximate length of day for t, the day of the year from March 21. Considering questions such as when to plant a garden (i.e., when there are at least 7 hours of midday sunlight), students might estimate that a 14-hour day is optimal. Students solve $f(t) = 14$ and find that May 1 and August 10 mark this interval of time.

- or for the frequency to be $\frac{1}{365}$ cycles/day



Students can investigate many other trigonometric modeling situations, such as simple predator–prey models, sound waves, and noise-cancellation models.
Source: UA Progressions Documents 2013c, 19.

Conceptual Category: Number and Quantity

Polynomial, Rational, and Radical Relationships

Perform arithmetic operations with complex numbers.

M.A2HS.1
Know there is a complex number i such that $i^2 = -1$, and every complex number has the form a + bi with a and b real.

M.A2HS.2
Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.

M.A2HS.3
Solve quadratic equations with real coefficients that have complex solutions. Instructional Note: Limit to polynomials with real coefficients.

M.A2HS.4
Extend polynomial identities to the complex numbers. For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$. Instructional Note: Limit to polynomials with real coefficients.

M.A2HS.5
Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials. Instructional Note: Limit to polynomials with real coefficients.

In Algebra I, students work with examples of quadratic functions and solve quadratic equations, encountering situations in which a resulting equation does not have a solution that is a real number— for example, $(x-2)^2 = -25$. In Algebra II, students complete their extension of the concept of *number* to include complex numbers, numbers of the form $a + bi$, where i is a number with the property that $i^2 = -1$. Students begin to work with complex numbers and apply their understanding of properties of operations (the commutative, associative, and distributive properties) and exponents and radicals to solve equations like those above, by finding square roots of negative numbers—for example, $\sqrt{-25} = \sqrt{25 \bullet (-1)} = \sqrt{25} \bullet \sqrt{-1} = 5i$ (**MHM7**). They also apply their understanding of properties of operations and exponents and radicals to solve equations:

$(x-2)^2 = -25$, which implies $|x-2| = 5i$, or $x = 2 \pm 5i$.

Equations like these have solutions, and the extended number system forms yet another system that behaves according to familiar rules and properties (**M.A2HS.1– M.A2HS.2; M.A2HS.3– M.A2HS.5**). By exploring examples of polynomials that can be factored with real and complex roots, students develop an understanding of the Fundamental Theorem of Algebra; they can show that the theorem is true for quadratic polynomials by an application of the quadratic formula and an understanding of the relationship between roots of a quadratic equation and the linear factors of the quadratic polynomial (**MHM2**).

Conceptual Category: Algebra

Along with the number and quantity standards in Algebra II, the algebra conceptual category standards develop the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multi-digit integers and division of polynomials with long division of integers. Rational numbers extend the arithmetic of integers by allowing division by all numbers except zero; similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme of this section is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Polynomial, Rational, and Radical Relationships
Interpret the structure of expressions. **M.A2HS.6** Interpret expressions that represent a quantity in terms of its context. a. Interpret parts of an expression, such as terms, factors, and coefficients. b. Interpret complicated expressions by viewing one or more of their parts as a single

entity. For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.

Instructional Note: Extend to polynomial and rational expressions.

M.A2HS.7

Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. Instructional Note: Extend to polynomial and rational expressions.

Write expressions in equivalent forms to solve problems.

M.A2HS.8

Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. For example, calculate mortgage payments. Instructional Note: Consider extending this standard to infinite geometric series in curricular implementations of this course description.

In Algebra II, students continue to pay attention to the meaning of expressions in context and interpret the parts of an expression by "chunking"—that is, viewing parts of an expression as a single entity (**M.A2HS.6– M.A2HS.7**). For example, their facility in using special cases of polynomial factoring allows them to fully factor more complicated polynomials:

$$x^4 - y^4 = (x^2)^2 - (y^2)^2 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y).$$

In a physics course, students may encounter an expression such as $L_0\sqrt{1 - \frac{v^2}{c^2}}$, which arises in the theory of special relativity. Students can see this expression as the product of a constant L_0 and a term that is equal to 1 when $v = 0$ and equal to 0 when $v = c$. Furthermore, they might be expected to see this mentally, without having to go through a laborious process of evaluation. This involves combining large-scale structure of the expression—a product of L_0 and another term—with the meaning of internal components such as $\frac{v^2}{c^2}$ (UA Progressions Documents 2013b, 4).

By examining the sums of examples of finite geometric series, students can look for patterns to justify why the equation for the sum holds: $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{(1-r)}$. They may derive the formula with proof by mathematical induction (**MHM3**) or by other means (**M.A2HS.8**), as shown in the following example.

Example: Sum of a Geometric Series	**M.A2HS.8**

Students should investigate several concrete examples of finite geometric series (with $r \neq 1$) and use spreadsheet software to investigate growth in the sums and patterns that arise (**MHM5, MHM8**).

Geometric series have applications in several areas, including calculating mortgage payments, calculating totals for annual investments such as retirement accounts, finding total payout amounts for lottery winners, and more (**MHM4**). In general, a finite geometric series has this form:

$$\sum_{k=0}^{n} ar^k = a(1 + r + r^2 + \ldots + r^{n-1} + r^n)$$

If the sum of this series is denoted by , then some algebraic manipulation shows that:

$$S - rS = a - ar^{n+1}.$$

Applying the distributive property to the common factors and solving for shows that:

$$S(1 - r) = a(1 - r^{n+1}),$$

so that:

$$S = \frac{a(1 - r^{n+1})}{1 - r}.$$

Students hone their ability to flexibly see expressions such as $A_n = A_0\left(1 + \frac{.15}{12}\right)^n$ as describing the total value of an investment at 15% interest, compounded monthly, for a number of compoundings, n. Moreover, they can interpret the following equation as a type of geometric series that would calculate the total value in an investment account at the end of one year if $100 is deposited at the beginning of each month (**MHM2, MHM4, MHM7**):

$$A_1 + A_2 + \ldots + A_{12} = 100\left(1 + \frac{.15}{12}\right)^1 + 100\left(1 + \frac{.15}{12}\right)^2 + \ldots + 100\left(1 + \frac{.15}{12}\right)^{12}$$

They apply the formula for geometric series to find this sum.

Polynomial, Rational, and Radical Relationships
Perform arithmetic operations on polynomials. **M.A2HS.9**

Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. Instructional Note: Extend beyond the quadratic polynomials found in Algebra I.

Understand the relationship between zeros and factors of polynomials.

M.A2HS.10

Know and apply the Remainder Theorem: For a polynomial p(x) and a number a, the remainder on division by x – a is p(a), so p(a) = 0 if and only if (x – a) is a factor of p(x).

M.A2HS.11

Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

M.A2HS.12

Prove polynomial identities and use them to describe numerical relationships. For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples. Instructional Note: This cluster has many possibilities for optional enrichment, such as relating the example in M.A2HS.10 to the solution of the system $u^2 + v^2 = 1$, $v = t(u+1)$, relating the Pascal triangle property of binomial coefficients to $(x + y)^{n+1} = (x + y)(x + y)^n$, deriving explicit formulas for the coefficients, or proving the binomial theorem by induction.

M.A2HS.13

Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle. Instructional Note: This cluster has many possibilities for optional enrichment, such as relating the example in M.A2HS.10 to the solution of the system $u^2 + v^2 = 1$, $v = t(u+1)$, relating the Pascal triangle property of binomial coefficients to $(x + y)^{n+1} = (x + y)(x + y)^n$, deriving explicit formulas for the coefficients, or proving the binomial theorem by induction.

Rewrite rational expressions.

M.A2HS.14

Rewrite simple rational expressions in different forms; write a(x)/b(x) in the form q(x) + r(x)/b(x), where a(x), b(x), q(x), and r(x) are polynomials with the degree of r(x) less than the degree of b(x), using inspection, long division, or, for the more complicated examples, a computer algebra system. Instructional Note: The limitations on rational functions apply to the rational expressions.

M.A2HS.15
Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions. Instructional Note: This standard requires the general division algorithm for polynomials.

In Algebra II, students continue to develop their understanding of the set of polynomials as a system analogous to the set of integers that exhibits particular properties, and they explore the relationship between the factorization of polynomials and the roots of a polynomial (**M.A2HS.9–M.A2HS.11**). It is shown that when a polynomial $p(x)$ is divided by $(x-a)$, $p(x)$ is written as $p(x) = q(x) \bullet (x-a) + r$, where r is a constant. This can be done by inspection or by polynomial long division (**M.A2HS.14**). It follows that $p(a) = q(a) \bullet (a-a) + r = q(a) \bullet 0 + r = r$, so that $(x-a)$ is a factor of $p(x)$ if and only if $p(a) = 0$. This result is generally known as the Remainder Theorem (**M.A2HS.10**), and provides an easy check to see if a polynomial has a given linear polynomial as a factor. This topic should not be simply reduced to "synthetic division," which reduces the theorem to a method of carrying numbers between registers, something easily done by a computer, while obscuring the reasoning that makes the result evident. It is important to regard the Remainder Theorem as a theorem, not a technique (**MHM3**) [UA Progressions Documents 2013b, 7].

Students use the zeros of a polynomial to create a rough sketch of its graph and connect the results to their understanding of polynomials as functions (**M.A2HS.11**). The idea that polynomials can be used to approximate other functions is important in higher mathematics courses such as Calculus, and standard **M.A2HS.11** is the first step in a progression that can lead students, as an extension topic, to constructing polynomial functions whose graphs pass through any specified set of points in the plane.

In Algebra II, students explore rational functions as a system analogous to that of rational numbers. They see rational functions as useful for describing many real-world situations—for instance, when rearranging the equation $d = rt$ to express the rate as a function of the time for a fixed distance d_0 and obtaining $r = \frac{d_0}{t}$. Now students see that any two polynomials can be divided in much the same way as with numbers (provided the divisor is not zero). Students first understand rational expressions as similar to other expressions in algebra, except that rational expressions have the form $\frac{a(x)}{b(x)}$ for both $a(x)$ and $b(x)$ polynomials. They should have opportunities to evaluate various rational expressions for many values of x, both by hand and using software, perhaps discovering that when the degree of $b(x)$ is larger than the degree of $a(x)$, the value of the expression gets smaller in absolute value as $|x|$ gets larger. When students understand the behavior of rational expressions in this way, it helps them see rational expressions as functions and sets the stage for working with simple rational functions.

Example: The Juice-Can Equations	**M.A2HS.14; M.A2HS.33**

If someone wanted to investigate the shape of a juice can of minimal surface area, the investigation could begin in the following way. If the volume V_0 is fixed, then the expression for the volume of the can is $V_0 = \pi r^2 h$, where h is the height of the can and r is the radius of the circular base. On the other hand, the surface area S is given by the following formula:

$$S = 2\pi rh + 2\pi r^2$$

This is because the two circular bases of the can contribute $2\pi r^2$ units of surface area, and the outside surface of the can contributes an area in the shape of a rectangle with length equal to the circumference of the base, $2\pi r$, and height equal to h. Since the volume is fixed, h can be found in terms of r: $h = \frac{V_0}{\pi r^2}$. Then this can be substituted into the equation for the surface area:

$$\begin{aligned} S &= 2\pi r \bullet \frac{V_0}{\pi r^2} + 2\pi r^2 \\ &= \frac{2V_0}{r} + 2\pi r^2 \end{aligned}$$

This equation expresses the surface area S as a (rational) function of r, which can then be analyzed.

(Also refer to standards **M.A2HS.26** and **M.A2HS.35**.)

In addition, students are able to rewrite rational expressions in the form $a(x) = q(x) \bullet b(x) + r(x)$, where $r(x)$ is a polynomial of degree less than $b(x)$, by inspection or by using polynomial long division. They can flexibly rewrite this expression as $\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}$ as necessary—for example, to highlight the end behavior of the function defined by the expression $\frac{a(x)}{b(x)}$. To make working with rational expressions more than just an exercise in the proper manipulation of symbols, instruction should focus on the characteristics of rational functions that can be understood by rewriting them in the ways described above (e.g., rates of growth, approximation, roots, axis intersections, asymptotes, end behavior, and so on).

Modeling with Functions

Create equations that describe numbers or relationships.

M.A2HS.23

Create equations and inequalities in one variable and use them to solve problems. Instructional Note: Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

M.A2HS.24
Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
Instructional Note: While functions will often be linear, exponential, or quadratic the types of problems should draw from more complex situations than those addressed in Algebra I. (e.g., Finding the equation of a line through a given point perpendicular to another line allows one to find the distance from a point to a line).

M.A2HS.25
Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. (e.g., Represent inequalities describing nutritional and cost constraints on combinations of different foods.) Instructional Note: While functions will often be linear, exponential, or quadratic the types of problems should draw from more complex situations than those addressed in Algebra I. For example, finding the equation of a line through a given point perpendicular to another line allows one to find the distance from a point to a line.

M.A2HS.26
Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. (e.g., Rearrange Ohm's law V = IR to highlight resistance R.) While functions will often be linear, exponential, or quadratic the types of problems should draw from more complex situations than those addressed in Algebra I. For example, finding the equation of a line through a given point perpendicular to another line allows one to find the distance from a point to a line. This example applies to earlier instances of this standard, not to the current course.

Students in Algebra II work with all available types of functions to create equations (**M.A2HS.23**). Although the functions referenced in standards **M.A2HS.24– M.A2HS.26** will often be linear, exponential, or quadratic, the types of problems should draw from more complex situations than those addressed in Algebra I. For example, knowing how to find the equation of a line through a given point perpendicular to another line makes it possible to find the distance from a point to a line. The Juice-Can Equation example presented previously in this section is connected to standard **M.A2HS.26**.

Polynomial, Rational, and Radical Relationships
Understand solving equations as a process of reasoning and explain the reasoning. **M.A2HS.16** Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise. Instructional Note: Extend to simple rational and radical equations. **Represent and solve equations and inequalities graphically.** **M.A2HS.17** Explain why the x-coordinates of the points where the graphs of the equations y = f(x) and y = g(x) intersect are the solutions of the equation f(x) = g(x); find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Instructional Note: Include cases where f(x) and/or g(x) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. Instructional Note: Include combinations of linear, polynomial, rational, radical, absolute value, and exponential functions.

Students extend their equation-solving skills to those involving rational expressions and radical equations; they make sense of extraneous solutions that may arise (**M.A2HS.16**). In particular, students understand that when solving equations, the flow of reasoning is generally forward, in the sense that it is assumed a number x is a solution of the equation and then a list of possibilities for x is found. However, not all steps in this process are reversible. For example, although it is true that if $x = 2$, then $x^2 = 4$, it is not true that if $x^2 = 4$, then $x = 2$, as $x = -2$ also satisfies this equation (UA Progressions Documents 2013b, 10). Thus students understand that some steps are reversible and some are not, and they anticipate extraneous solutions. In addition, students continue to develop their understanding of solving equations as solving for values of x such that $f(x) = g(x)$, now including combinations of linear, polynomial, rational, radical, absolute value, and exponential functions (**M.A2HS.17**). Students also understand that some equations can only be solved approximately with the tools they possess.

Conceptual Category: Geometry

No traditional Algebra II course would be complete without an examination of planar curves represented by the general equation $ax^2 + by^2 + cx + dy + e = 0$. In Algebra II, students use "completing the square" (a skill learned in Algebra I) to decide if the equation represents a circle or parabola. They graph the shapes and relate the graph to the equation. The study of ellipses and hyperbolas is reserved for a later course.

Conceptual Category: Statistics and Probability

Students in Algebra II move beyond analysis of data to make sound statistical decisions based on probability models. The reasoning process is as follows: develop a statistical question in the form of a hypothesis (supposition) about a population parameter, choose a probability model for collecting data relevant to that parameter, collect data, and compare the results seen in the data with what is expected under the hypothesis. If the observed results are far from what is expected and have a low probability of occurring under the hypothesis, then that hypothesis is called into question. In other words, the evidence against the hypothesis is weighed by probability (**M.A2HS.38– M.A2HS.39**) [UA Progressions Documents 2012d]. By investigating simple examples of simulations of experiments and observing outcomes of the data, students gain an understanding of what it means for a model to fit a particular data set (**M.A2HS.39**). This includes comparing theoretical and empirical results to evaluate the effectiveness of a treatment.

Inferences and Conclusions from Data
Summarize, represent, and interpret data on a single count or measurement variable.
M.A2HS.37 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. Instructional Note: While students may have heard of the normal distribution, it is unlikely that they will have prior experience using it to make specific estimates. Build on students' understanding of data distributions to help them see how the normal distribution uses area to make estimates of frequencies (which can be expressed as probabilities). Emphasize that only some data are well described by a normal distribution.

Although students may have heard of the normal distribution, it is unlikely that they will have prior experience using the normal distribution to make specific estimates. In Algebra II, students build on their understanding of data distributions to help see how the normal distribution uses area to make estimates of frequencies (which can be expressed as probabilities). It is important for students to see that only some data are well described by a normal distribution (**M.A2HS.37**). In addition, they can learn through examples the *empirical rule*: that for a normally distributed data set, 68% of the data lie within one standard deviation of the mean and 95% within two standard deviations of the mean.

Example: The Empirical Rule
M.A2HS.37

Suppose that SAT mathematics scores for a particular year are approximately normally distributed, with a mean of 510 and a standard deviation of 100.
a. What is the probability that a randomly selected score is greater than 610?
b. What is the probability that a randomly selected score is greater than 710?
c. What is the probability that a randomly selected score is between 410 and 710?
d. If a student's score is 750, what is the student's percentile score (the proportion of scores below 750)?

Solutions:
a. The score 610 is one standard deviation above the mean, so the tail area above that is about half of 0.32, or 0.16. The calculator gives 0.1586.
b. The score 710 is two standard deviations above the mean, so the tail area above that is about half of 0.05, or 0.025. The calculator gives 0.0227.
c. The area under a normal curve from one standard deviation below the mean to two standard deviations above the mean is about 0.815. The calculator gives 0.8186.
d. Using either the normal distribution given or the standard normal (for which 750 translates to a z-score of 2.4), the calculator gives 0.9918.

Inferences and Conclusions from Data

Understand and evaluate random processes underlying statistical experiments.

M.A2HS.38
Understand statistics as a process for making inferences about population parameters based on a random sample from that population. Instructional Note: Include comparing theoretical and empirical results to evaluate the effectiveness of a treatment.

M.A2HS.39
Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. (e.g., A model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?)

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

M.A2HS.40
Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. Instructional Note: In earlier grades, students are introduced to different ways of collecting data and use

graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data is collected determines the scope and nature of the conclusions that can be drawn from that data. The concept of statistical significance is developed informally through simulation as meaning a result that is unlikely to have occurred solely as a result of random selection in sampling or random assignment in an experiment.

M.A2HS.41

Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. Instructional Note: In earlier grades, students are introduced to different ways of collecting data and use graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data is collected determines the scope and nature of the conclusions that can be drawn from that data. The concept of statistical significance is developed informally through simulation as meaning a result that is unlikely to have occurred solely as a result of random selection in sampling or random assignment in an experiment. Focus on the variability of results from experiments—that is, focus on statistics as a way of dealing with, not eliminating, inherent randomness.

M.A2HS.42

Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. Instructional Note: In earlier grades, students are introduced to different ways of collecting data and use graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data is collected determines the scope and nature of the conclusions that can be drawn from that data. The concept of statistical significance is developed informally through simulation as meaning a result that is unlikely to have occurred solely as a result of random selection in sampling or random assignment in an experiment. Focus on the variability of results from experiments—that is, focus on statistics as a way of dealing with, not eliminating, inherent randomness.

M.A2HS.43

Evaluate reports based on data. Instructional Note: In earlier grades, students are introduced to different ways of collecting data and use graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data is collected determines the scope and nature of the conclusions that can be drawn from that data. The concept of statistical significance is developed informally through simulation as meaning a result that is unlikely to have occurred solely as a result of random selection in sampling or random assignment in an experiment.

In earlier grade levels, students are introduced to different ways of collecting data and use graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data are collected determines the scope and nature of the conclusions that can be drawn from those data. The concept of *statistical significance* is developed informally through simulation as meaning a result that is unlikely to have occurred solely through random selection in sampling or random assignment in an experiment (NGA/CCSSO 2010a). When covering standards **M.A2HS.41– M.A2HS.42**, instructors should focus on the variability of results from experiments—that is, on statistics as a way of handling, not eliminating, inherent randomness. Because standards **M.A2HS.38– M.A2HS.43** are all modeling standards, students should have ample opportunities to explore statistical experiments and informally arrive at statistical techniques.

Example: Estimating a Population Proportion **M.A2HS.38– M.A2HS.43**

Suppose a student wishes to investigate whether 50% of homeowners in her neighborhood will support a new tax to fund local schools. If she takes a random sample of 50 homeowners in her neighborhood, and 20 support the new tax, then the *sample proportion* agreeing to pay the tax would be 0.4. But is this an accurate measure of the true proportion of homeowners who favor the tax? How can this be determined?

If this sampling situation (**MHM4**) is simulated with a graphing calculator or spreadsheet software under the assumption that the true proportion is 50%, then the student can arrive at an understanding of the *probability* that her randomly sampled proportion would be 0.4. A simulation of 200 trials might show that 0.4 arose 25 out of 200 times, or with a probability of 0.125. Thus, the chance of obtaining 40% as a sample proportion is not insignificant, meaning that a true proportion of 50% is plausible.

Adapted from UA Progressions Documents 2012d.

Inferences and Conclusions from Data

Use probability to evaluate outcomes of decisions.

M.A2HS.44

Use probabilities to make fair decisions (e.g., drawing by lots or using a random number generator). Instructional Note: Extend to more complex probability models. Include situations such as those involving quality control, or diagnostic tests that yield both false positive and false negative results.

M.A2HS.45

Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, and/or pulling a hockey goalie at the end of a game). Instructional Note: Extend to more complex probability models. Include situations such as those involving quality control, or diagnostic tests that yield both false positive and false negative results.

As in Geometry, students apply probability models to make and analyze decisions. In Algebra II, this skill is extended to more complex probability models, including situations such as those involving quality control or diagnostic tests that yield both false-positive and false-negative results.

Algebra II is the culmination of the three-course sequence in the Traditional Pathway for mathematics. Students completing this pathway will be well prepared for advanced mathematics and should be encouraged to continue their study of mathematics with Trigonometry/Pre-calculus or other mathematics courses from the Fourth Course Options listed in the West Virginia College- and Career Readiness Standards, dual credit mathematics courses or courses offered through WV Virtual School. Policy 2510 requires that, beginning with the 2018-2019 freshman cohort, all students must be enrolled in a mathematics course during each year of high school.

Common Misconceptions – By Domain

Number and Quantity

- Students often have difficulty differentiating between finding the solution to an equation such as $x^2 = 25$ and taking the square root of a number. The equation $x^2 = 25$ has two solutions, 5 and -5. Expressions such as $\sqrt{3^2}$ or $\sqrt{(-7)^2}$ ask for the "principal" square root. As a result, $\sqrt{3^2}$ = 3 and $\sqrt{(-7)^2}$ = 7

- Student confusion in differentiating between expressions involving the square of a negative number and the negative of a square, often extend to square roots. Students may need help in appreciating the Order of Operations in differentiating between the expressions.

$(-5)^2$ → (-5)(-5) → 25

$-(5)^2$ → - (5)(5) → -25

$\sqrt{(-5)^2}$ → $\sqrt{25}$ → 5

$-\sqrt{5^2}$ → - $\sqrt{25}$ → -5

$\sqrt{(5)^2}$ → $\sqrt{-25}$ → 5

- Students may have difficulty when negative exponents involve fractions. Students may incorrectly believe that $16^{-\frac{1}{2}}$ means -4 rather than $\frac{1}{4}$.

$$16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$$

Algebra

- Students may have difficulty distinguishing between the terms of an expression and the variable of an expression. For example, students may reason that the trinomial $3a + b + 5cd$ has four terms because the student is counting variable rather than terms. Often students do not recognize a constant as a term and incorrectly identify the trinomial $4x - 2y + 7$ as having two terms.

- Students who rely solely on procedures may believe they need multiply the binomials in an equation such as $(x + 4)(x - 3) = 0$ and then factor the resulting expression to solve for the zeros. Students need to develop an understanding of the concept of the Zero Factor Property. Students may experience similar confusion in applying the Zero Factor Property in solving equations such as $5(x - 7)(x + 1) = 0$ or $6x(x - 7) = 0$.

- Students who mistakenly believe that $\sqrt{x^2 + y^2}$ is equivalent to $x + y$, may deduce this from the misconception that $(x + y)^2$ is equivalent to $x^2 + y^2$.

- Students who mistakenly equate $(x + y)^2$ with $x^2 + y^2$ or equate $(x - y)^2$ with $x^2 - y^2$ have misconceptions regarding the concept of finding the square of a number or expression. These misconceptions may stem from students' prior difficulty in recognizing that $(-5)^2$ → (-5)(-5) →25 or that $-(5)^2$ → –(5)(5) → -25.

- Students who incorrectly equate rational expressions such as $\frac{x^2 - 6x + 9}{x - 3}$ with $x - 2x + 9$ or $x - 6x - 3$ or $x^2 + 2x + 9$, etc., have difficulty distinguishing between terms and factors.

- Students may demonstrate difficulties in solving the equation $\sqrt{x - 1}$ = $x - 7$. Students may appropriately decide to square both sides of the equation and write $(\sqrt{x - 1}\,)^2$ = $(x - 7)^2$ Student error may arise in writing $(\sqrt{x - 1}\,)^2$ = x^2 - 7^2, demonstrating misconceptions in squaring binomials (or multiplying) binomials.

- The procedure for solving equations and inequalities are so similar. As a result, students often forget to attend to precision when multiplying and dividing by a negative number when solving inequalities. Students may correctly solve equations such as $-2x = 6$ and find its solution to be $x = -3$. Misconceptions arise when students equate solving the equation with a similar inequality $-2x \leq 6$ and incorrectly determine that $x \leq -3$. The misconception can be addressed by encouraging students to verify solutions to equations and to inequalities.

- Students often solve rational and radical equations without checking to determine if any solution may be erroneous. Students should be encouraged to verify solutions.

Functions

- Students may erroneously interpret the notation $g(3)$ to mean "$g\ times\ 3$".

- Students often believe that all functions must use the symbols $f, x,$ and y.

- When graphing students may confuse the parts of the slope-equation form of a linear equation. Students may incorrectly determine that the function $g(x) = 2x + \frac{3}{5}$ has a slope of $\frac{3}{5}$ and a y-intercept of 2. Students may be relying on a procedural understanding that the y-intercept is always an integer and the slope or $\frac{rise}{run}$ must be a fraction.

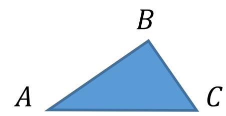
- Students often incorrectly assume that the function $f(x + k)$ where $k > 0$ will result in a horizontal shift of the graph k units to the right.

Geometry

- Students often think of congruence as "figures with the same shape and size." While this understanding is not incorrect, it is important to that the continually emphasize the link of congruence with rigid motions and show that rigid motions do in fact produce "figures with the same shape and size."

- Students may look at scale factor as the distance that is added on to the original distance. Dilations where the center of dilation is a vertex of a figure can prove challenging because the sides of the pre-image and image overlap.

- Students may have difficulty identifying the relevant sides in relationship to a given angle, especially If the triangle is not depicted in a typical position where its legs are horizontal and vertical. For example, given $\Delta\, ABC$ below with right angle B, students may struggle to understand that $\sin A = \frac{BC}{AC}$ and that $\sin A \neq \frac{BC}{AB}$.



- Students may have difficulty differentiating between the meaning of the statements $\Delta\, ABC \cong \Delta\, DEF$ and $\Delta\, ABC \sim \Delta\, DEF$. While $\Delta\, ABC \cong \Delta\, DEF$ implies that AB = DE, $\Delta\, ABC \sim \Delta\, DEF$ does not imply that AB = DE.

- Students often have difficulty differentiating among $\overleftarrow{EF}$, $\overrightarrow{EF}$, $\overleftrightarrow{EF}$, and $\overline{EF}$.

Statistics and Probability

- Students often use the word *outlier* inaccurately, failing to verify that it satisfies the necessary conditions. Using terms such as *unusual feature* or *data point* can help students avoid using the term outlier when it is not appropriate.

- Students commonly believe that any data that is collected should follow a normal distribution.

- Students need to understand that association does not necessarily provide evidence for cause and effect.

Steven L. Paine, Ed.D.
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