West Virginia math4life

Educators' Guide for Mathematics

Geometry

West Virginia DEPARTMENT OF EDUCATION

Geometry

The fundamental purpose of the Geometry course is to introduce students to formal geometric proofs and the study of plane figures, culminating in the study of right-triangle trigonometry and circles. Students begin to formally prove results about the geometry of the plane by using previously defined terms and notions. Similarity is explored in greater detail, with an emphasis on discovering trigonometric relationships and solving problems with right triangles. The correspondence between the plane and the Cartesian coordinate system is explored when students connect algebra concepts with geometry concepts. Students explore probability concepts and use probability in real-world situations. The major mathematical ideas in the Geometry course include geometric transformations, proving geometric theorems, congruence and similarity, analytic geometry, right-triangle trigonometry, and probability.

The standards in the traditional Geometry course come from the following conceptual categories: modeling, geometry, and statistics and probability. The content of the course is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Geometry

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). In the higher mathematics courses, students begin to formalize their geometry experiences from elementary and middle school, using definitions that are more precise and developing careful proofs. The standards for grades seven and eight call for students to see two-dimensional shapes as part of a generic plane (i.e., the *Euclidean plane*) and to explore transformations of this plane as a way to determine whether two shapes are congruent or similar. These concepts are formalized in the Geometry course, and students use transformations to prove geometric theorems. The definition of *congruence* in terms of rigid motions provides a broad understanding of this means of proof, and students explore the consequences of this definition in terms of congruence criteria and proofs of geometric theorems.

Students investigate triangles and decide when they are similar—and with this newfound knowledge and their prior understanding of proportional relationships, they define trigonometric ratios and solve problems by using right triangles. They investigate circles and prove theorems about them. Connecting to their prior experience with the coordinate plane, they prove geometric theorems by using coordinates and describe shapes with equations.

Students extend their knowledge of area and volume formulas to those for circles, cylinders, and other rounded shapes. Finally, continuing the development of statistics and probability, students investigate probability concepts in precise terms, including the independence of events and conditional probability.

Examples of Key Advances from Previous Grade Levels or Courses

- Because concepts such as *rotation, reflection,* and *translation* are treated in the grade-eight standards mostly in the context of hands-on activities and with an emphasis on geometric intuition, the Geometry course places equal weight on precise definitions.
- In kindergarten through grade eight, students work with a variety of geometric measures: length, area, volume, angle, surface area, and circumference. In Geometry, students apply these component skills in tandem with others in the course of modeling tasks and other substantial applications (**MHM4**).
- The skills that students develop in Algebra I relevant to simplifying and transforming square roots will be useful when solving problems that involve distance or area and that make use of the Pythagorean Theorem.
- Students in grade eight learn the Pythagorean Theorem and use it to determine distances in a coordinate system (**M.8.21–M.8.23**). In Geometry, students build on their understanding of distance in coordinate systems and draw on their growing command of algebra to connect equations and graphs of circles (**M.GHS.39**).
- The algebraic techniques developed in Algebra I can be applied to study analytic geometry. Geometric objects can be analyzed by the algebraic equations that give rise to them. Algebra can be used to prove some basic geometric theorems in the Cartesian plane.

Connecting Mathematical Habits of Mind and Content

The Mathematical Habits of Mind (**MHM**) apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a rigorous, coherent, useful, and logical subject. The Mathematical Habits of Mind represent a picture of what it looks like for students to *do mathematics* and, to the extent possible, content instruction should include attention to appropriate practice standards. The Geometry course offers ample opportunities for students to engage with each Mathematical Habit of Mind. The following table offers some examples.

Mathematical Habits of Mind—Explanation and Examples for Geometry

Mathematical Habits of Mind	Explanation and Examples
HM1 Make sense of problems and persevere in solving them.	Students construct accurate diagrams of geometry problems to help make sense of them. They organize their work so that others can follow their reasoning (e.g., in proofs).
MHM2 Reason abstractly and quantitatively.	Students understand that the coordinate plane can be used to represent geometric shapes and transformations, and therefore they connect their understanding of number and algebra to geometry.
MHM3 Construct viable arguments and critique the reasoning of others.	Students reason through the solving of equations, recognizing that solving an equation involves more than simply following rote rules and steps. They use language such as "If ________, then ________" when explaining their solution methods and provide justification for their reasoning.
MHM4 Model with mathematics.	Students apply their new mathematical understanding to real-world problems. They learn how transformational geometry and trigonometry can be used to model the physical world.
MHM5 Use appropriate tools strategically.	Students make use of visual tools for representing geometry, such as simple patty paper, transparencies, or dynamic geometry software.
MHM6 Attend to precision.	Students develop and use precise definitions of geometric terms. They verify that a particular shape has specific properties and justify the categorization of the shape (e.g., a rhombus versus a quadrilateral).
MHM7 Look for and make use of structure.	Students construct triangles in quadrilaterals or other shapes and use congruence criteria of triangles to justify results about those shapes.
MHM8 Look for and express regularity in repeated reasoning.	Students explore rotations, reflections, and translations, noticing that some attributes of shapes (e.g., parallelism, congruency, orientation) remain the same. They develop properties of transformations by generalizing these observations.

MHM4 holds a special place throughout the higher mathematics curriculum, as modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a high place in instruction.

Geometry Content Standards

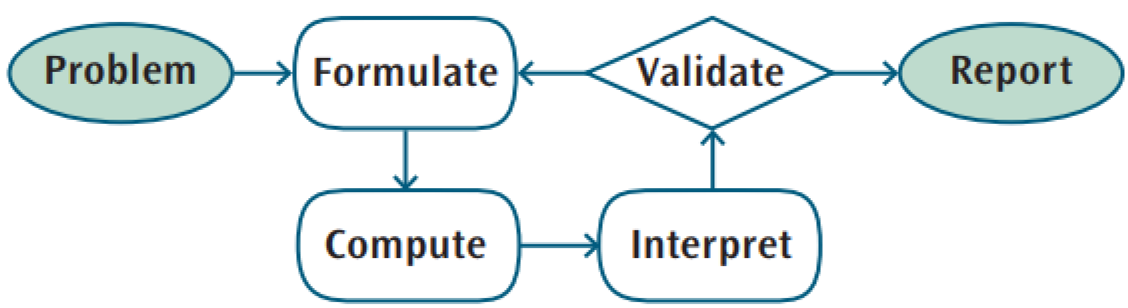
The Geometry course is organized by domains, clusters, and standards. The overall purpose and progression of the standards included in the Geometry course are described below, according to each conceptual category. Standards that are considered new for teachers of secondary-grades are discussed more thoroughly than other standards.

Conceptual Category: Modeling

Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them, and the mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise: *Which of the quantities present in this situation are known and which are unknown? What can I generalize? Is there some way to introduce into this diagram a known shape that gives more information?* Students need to decide on a solution path, which may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They validate their work by moving between calculations done by hand and software-assisted computations.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, and the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see the diagram below of a model cycle. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Modeling Cycle



The examples for the course are framed as much as possible to illustrate the concept of mathematical modeling. The important ideas of proving geometric theorems, congruence and similarity, analytic geometry, right-triangle trigonometry, and probability can be explored in this way.

Conceptual Category: Geometry

A large portion of instruction in the traditional Geometry course is formed by the standards of the geometry conceptual category. Here, students develop the ideas of congruence and similarity through transformations. They prove theorems, both with and without the use of coordinates. They explore right-triangle trigonometry, as well as circles and parabolas. **MHM3,** “Construct viable arguments and critique the reasoning of others,” plays a predominant role throughout the Geometry course.

Congruence, Proof and Constructions

Experiment with transformations in the plane.

M.GHS.1

Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

M.GHS.2

Represent transformations in the plane using, for example, transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch). Instructional Note: Build on student experience with rigid motions from earlier grades. Point out the basis of rigid motions in geometric concepts, (e.g., translations move points a specified distance along a line parallel to a specified line; rotations move objects along a circular arc with a specified center through a specified angle).

M.GHS.3

Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself. Instructional Note: Build on student experience with rigid motions from earlier grades. Point out the basis of rigid motions in geometric concepts, (e.g., translations move points a specified distance along a line parallel to a specified line; rotations move objects along a circular arc with a specified center through a specified angle).

M.GHS.4
Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments. Instructional Note: Build on student experience with rigid motions from earlier grades. Point out the basis of rigid motions in geometric concepts (e.g., translations move points a specified distance along a line parallel to a specified line; rotations move objects along a circular arc with a specified center through a specified angle).

M.GHS.5
Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, for example, graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another. Instructional Note: Build on student experience with rigid motions from earlier grades. Point out the basis of rigid motions in geometric concepts, (e.g., translations move points a specified distance along a line parallel to a specified line; rotations move objects along a circular arc with a specified center through a specified angle)

Understand congruence in terms of rigid motions.

M.GHS.6
Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent. Instructional Note: Rigid motions are at the foundation of the definition of congruence. Students reason from the basic properties of rigid motions (that they preserve distance and angle), which are assumed without proof. Rigid motions and their assumed properties can be used to establish the usual triangle congruence criteria, which can then be used to prove other theorems.

M.GHS.7
Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent. Instructional Note: Rigid motions are at the foundation of the definition of congruence. Students reason from the basic properties of rigid motions (that they preserve distance and angle), which are assumed without proof. Rigid motions and their assumed properties can be used to establish the usual triangle congruence criteria, which can then be used to prove other theorems.

M.GHS.8
Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions. Instructional Note: Rigid motions are at the foundation of the definition of congruence. Students reason from the basic properties of rigid motions (that

they preserve distance and angle), which are assumed without proof. Rigid motions and their assumed properties can be used to establish the usual triangle congruence criteria, which can then be used to prove other theorems.

In geometry, the commonly held (but imprecise) definition that shapes are congruent when they "have the same size and shape" is replaced by a more mathematically precise definition (**MHM6**): *Two shapes are congruent if there is a sequence of rigid motions in the plane that takes one shape exactly onto the other*. This definition is explored intuitively in the grade-eight standards, but in the Geometry course it is investigated more closely. In grades seven and eight, students experiment with transformations in the plane; however, the Geometry course requires that students build more precise definitions for the *rigid motions* (rotation, reflection, and translation) based on previously defined and understood terms such as *point, line, between, angle, circle, perpendicular,* and so forth (**M.GHS.1, MGHS.3, M.GHS.4**). Students base their understanding of these definitions on their experience with transforming figures using patty paper, transparencies, or geometry software (**M.GHS.2, M.GHS.3, M.GHS.5; MHM5**). Students were first introduced to these definitions in grade eight. These transformations should be investigated both in a general plane as well as on a coordinate system—especially when transformations are explicitly described by using precise names of points, translation vectors, and lines of symmetry or reflection.

Example: Defining Rotations	**M.GHS.4**
Mrs. B wants to help her class understand the following definition of a *rotation:* A *rotation* about a point P through angle α is a transformation $A \longmapsto A'$ such that (1) if point A is different from P, then $PA = PA'$ and the measure of $\angle APA' = \alpha$; and (2) if point A is the same as point P, then $A' = A$. Mrs. B gives her students a handout with several geometric shapes on it and a point, P, indicated on the page. In pairs, students copy the shapes onto a transparency sheet and rotate them through various angles about P; then they transfer the rotated shapes back onto the original page and measure various lengths and angles as indicated in the definition. While justifying that the properties of the definition hold for the shapes given to them by Mrs. B, the students also make some observations about the effects of a rotation on the entire plane. For example:	

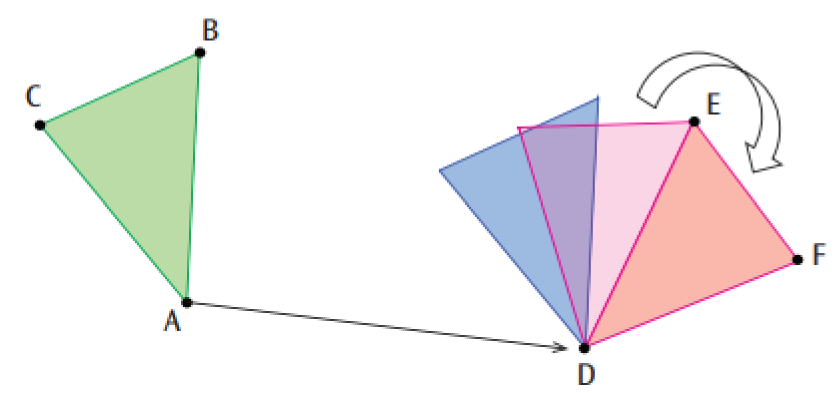
- Rotations preserve lengths.
- Rotations preserve angle measures.
- Rotations preserve parallelism.

In a subsequent exercise, Mrs. B plans to allow students to explore more rotations on dynamic geometry software, asking them to create a geometric shape and rotate it by various angles about various points, both part of the object and not part of the object.

In standards **M.GHS.6–M.GHS.8,** geometric transformations are given a more prominent role in the higher mathematics geometry curriculum than perhaps ever before. The new definition of *congruence* in terms of rigid motions applies to any shape in the plane, whereas previously, congruence seemed to depend on criteria that were specific only to particular shapes. For example, the side–side–side (SSS) congruence criterion for triangles did not extend to quadrilaterals, which seemed to suggest that congruence was a notion dependent on the shape that was considered. Although it is true that there are specific criteria for determining congruence of certain shapes, the basic notion of congruence is the same for all shapes. In the West Virginia College- and Career- Readiness Standards, the SSS criterion for triangle congruence is a *consequence* of the definition of congruence, just as the fact that if two polygons are congruent, then their sides and angles can be put into a correspondence such that each corresponding pair of sides and angles is congruent. This concept comprises the content of standards **M.GHS.7** and **M.GHS.8,** which derive congruence criteria for triangles from the new definition of congruence.

Standard **M.GHS.7** explicitly states that students show that two triangles are congruent *if and only if* corresponding pairs of sides and corresponding pairs of angles are congruent (**MHM3**). The depth of reasoning here is fairly substantial, as students must be able to show, using rigid motions, that congruent triangles have congruent corresponding parts and that, conversely, if the corresponding parts of two triangles are congruent, then there is a sequence of rigid motions that takes one

Illustration of the Reasoning That Congruent Corresponding Parts Imply Triangle Congruence



Point A is translated to D, the resulting image of ΔABC is rotated so as to place B onto E, and the image is then reflected along line segment DE to match point C to F.

triangle to the other. The second statement may be more difficult to justify than the first for most students, so a justification is presented here. Suppose there are two triangles ΔABC and ΔDEF such that the correspondence $A \leftrightarrow D$,$B \leftrightarrow E$, $C \leftrightarrow F$ results in pairs of sides and pairs of angles being congruent. If one triangle were drawn on a fixed piece of paper and the other drawn on a separate transparency, then a student could illustrate a translation, T, that takes point A to point D. A simple rotation R about point A, if necessary, takes point B to point E, which is certain to occur because $\overline{AB} \cong \overline{DE}$ and rotations preserve lengths. A final step that may be needed is a reflection S about the side AB, to take point C to point F. It is important to note why the image of point C is actually F. Since $\angle A$ is reflected about line $\overleftrightarrow{AB}$, its measure is preserved. Therefore, the image of side $\overline{AC}$ at least lies on line $\overline{DF}$, since $\angle A \cong \angle D$. But since $\overline{AC} \cong \overline{DF}$, it must be the case that the image of point C coincides with F. The previous discussion amounts to the fact that the sequence of rigid motions, T, followed by R, followed by S, maps ΔABC exactly onto ΔDEF. Therefore, if it is known that the corresponding parts of two triangles are congruent, then there is a sequence of rigid motions carrying one onto the other; that is, they are congruent. The informal proof presented here should be accessible to students in the Geometry course; see the accompanying illustration.

Similar reasoning applies for standard **M.GHS.8,** in which students justify the typical triangle congruence criteria such as ASA, SAS, and SSS. Experimentation with transformations of triangles where only two of the criteria are satisfied will result in counterexamples, and geometric constructions of triangles of prescribed side lengths (e.g., in the case of SSS) will leave little doubt that any triangle constructed with these side lengths will be congruent to another, and therefore that SSS holds (**MHM7**).

Congruence, Proof and Constructions

Prove geometric theorems.

M.GHS.9

Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints. Instructional Note: Encourage multiple ways of writing proofs, such as in narrative paragraphs, using flow diagrams, in two-column format, and using diagrams without words. Students should be encouraged to focus on the validity of the underlying reasoning while exploring a variety of formats for expressing that reasoning.

M.GHS.10

Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point. Instructional Note: Encourage multiple ways of writing proofs, such as in narrative paragraphs, using flow diagrams, in two-column format, and using diagrams without words. Students should be encouraged to focus on the validity of the underlying reasoning while exploring a variety of formats for expressing that reasoning. Implementation of this standard may be extended to include concurrence of perpendicular bisectors and angle bisectors as preparation for M.GHS.36.

M.GHS.11
Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals. Instructional Note: Encourage multiple ways of writing proofs, such as in narrative paragraphs, using flow diagrams, in two-column format, and using diagrams without words. Students should be encouraged to focus on the validity of the underlying reasoning while exploring a variety of formats for expressing that reasoning.

Make geometric constructions.

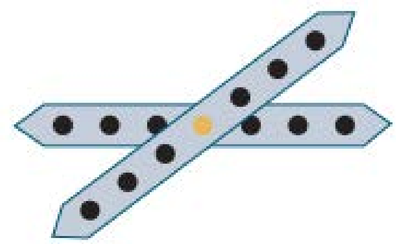
M.GHS.12
Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line. Instructional Note: Build on prior student experience with simple constructions. Emphasize the ability to formalize and explain how these constructions result in the desired objects. Some of these constructions are closely related to previous standards and can be introduced in conjunction with them.

M.GHS.13
Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle. Instructional Note: Build on prior student experience with simple constructions. Emphasize the ability to formalize and explain how these constructions result in the desired objects. Some of these constructions are closely related to previous standards and can be introduced in conjunction with them.

It is important to note that when the triangle criteria for congruence are established, students can begin to prove geometric theorems. Examples of such theorems are listed in standards **M.GHS.9–M.GHS.11.** The triangle congruence criteria are established results that can be used to prove new results. Instructors are encouraged to use a variety of strategies for engaging students to understand and write proofs, such as using numerous pictures to demonstrate results and generate strategies; using patty paper, transparencies, or dynamic geometry software to explore the steps in a proof; creating flowcharts and other organizational diagrams for outlining a proof; and writing step-by-step or paragraph formats for a completed proof (**MHM5**). Above all else, instructors should emphasize the reasoning involved in connecting one step in the logical argument to the next. Students should be encouraged to make conjectures based on experimentation, to justify their conjectures, and to communicate their reasoning to their peers (**MHM3**). The following example illustrates how students can be encouraged to experiment and construct hypotheses based on their observations.

Example: The Kite Factory	**M.GHS.11**

Kite engineers want to know how the shape of a kite—the length of the rods, where they are attached, the angle at which the rods are attached, and so on—affects how the kite flies. In this activity, students are given pieces of cardstock of various lengths, hole-punched at regular intervals so they can be attached in different places.



These two “rods” form the frame for a kite at the kite factory. By changing the angle at which the sticks are held and the places where the sticks are attached, students discover different properties of quadrilaterals.

Students are challenged to make conjectures and use precise language to describe their findings about which diagonals result in which quadrilaterals. They can discover properties unique to certain quadrilaterals, such as the fact that diagonals that are perpendicular bisectors of each other imply the quadrilateral is a rhombus. To see videos of this lesson being implemented in a high school classroom, visit http://www.insidemathematics.org/ (accessed March 26, 2015).

Similarity, Proof, and Trigonometry

Understand similarity in terms of similarity transformations.

M.GHS.14

Verify experimentally the properties of dilations given by a center and a scale factor.

a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

M.GHS.15

Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

M.GHS.16
Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

Prove theorems involving similarity.

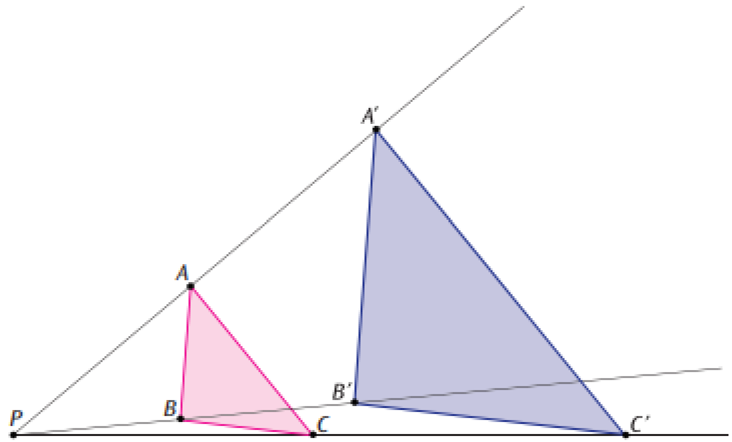
M.GHS.17
Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.

M.GHS.18
Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Because right triangles and triangle relationships play such an important role in applications and future mathematics learning, they are given a prominent role in the geometry conceptual category. A discussion of *similarity* is necessary first—and again, a more precise mathematical definition of similarity is given in the higher mathematics standards. Students work with *dilations* as a transformation in the grade-eight standards; now they explore the properties of dilations in more detail and develop an understanding of the notion of *scale factor* (**M.GHS.14**). Whereas it is common to say that objects that are similar have "the same shape," the new definition for two objects being similar is that there is a sequence of *similarity transformations*—translation, rotation, reflection, or dilation—that maps one object exactly onto the other. Standards **M.GHS.15** and **M.GHS.16** call for students to explore the consequences of two triangles being similar: that they have congruent angles and that their side lengths are in the same proportion. This new understanding gives rise to more results that are encapsulated in standards **M.GHS.17** and **M.GHS.18**.

Example: Experimenting with Dilations	**M.GHS.15–M.GHS.16**
Students are given opportunities to experiment with dilations and determine how they affect planar objects. Students first make sense of the definition of a *dilation of scale factor* $k > 0$ *with center* P as the transformation that moves a point A along the ray $\overline{PA}$ to a new point A', so that $\lvert\overline{PA'}\rvert = k \bullet \lvert\overline{PA}\rvert$. For example, using a ruler, students apply the dilation of scale factor 2.5 with center P to the points A, B, and C illustrated below. Once this is done, the students consider the two triangles ΔABC and $\Delta A'B'C'$, and they discover that the lengths of the corresponding sides of the triangles have the same ratio dictated by the scale factor (**M.GHS.15**). Students learn that parallel lines are taken to parallel lines by dilations; thus corresponding	

segments of ΔABC and $\Delta A'B'C'$ are parallel. After students have proved results about parallel lines intersected by a transversal, they can deduce that the angles of the triangles are congruent. Through experimentation, they see that the congruence of corresponding angles is a necessary and sufficient condition for the triangles to be similar, leading to the AA criterion for triangle similarity (**M.GHS.16**).



For a simple investigation, students can observe how the distance at which a projector is placed from a screen affects the size of the image on the screen (**MHM4**).

Similarity, Proof, and Trigonometry

Define trigonometric ratios and solve problems involving right triangles.

M.GHS.19
Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

M.GHS.20
Explain and use the relationship between the sine and cosine of complementary angles.

M.GHS.21
Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

Once the angle–angle (AA) similarity criterion for triangles is established, it follows that any two *right* triangles ΔABC and ΔDEF are similar when at least one pair of angles are congruent (say $\angle A \cong \angle D$), since the right angles are obviously congruent (say $\angle B \cong \angle E$). By similarity, the corresponding sides of the triangles are in proportion:

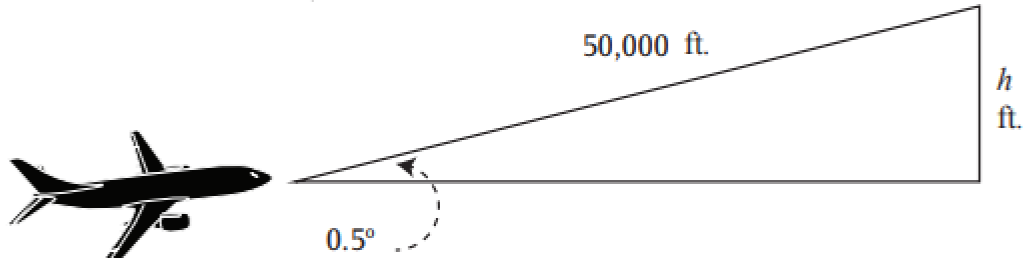
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Notice the first and third expressions in the statement of equality above can be rearranged to yield that:

$$\frac{AB}{AC} = \frac{DE}{DF}$$

Since the triangles in question are arbitrary, this implies that for any right triangle with an angle congruent to $\angle A$, the ratio of the side opposite to $\angle A$ and the hypotenuse of the triangle is a certain constant. This allows us to define unambiguously the *sine* of $\angle A$, denoted by $\sin A$, as the value of this ratio. In this way, students come to understand the trigonometric functions as relationships completely determined by angles (**M.GHS.19**). They further their understanding of these functions by investigating relationships between sine, cosine, and tangent; by exploring the relationship between the sine and cosine of complementary angles; and by applying their knowledge of right triangles to real-world situations (**MHM4**), such as in the example below (**M.GHS.19–M.GHS.21**). Experience working with many different triangles, finding their measurements, and computing ratios of the measurements found will help students understand the basics of the trigonometric functions.

Example: Using Trigonometric Relationships	**M.GHS.19–21**
Airplanes that travel at high speeds and low elevations often have onboard radar systems to detect possible obstacles in their path. The radar can determine the range of an obstacle and the angle of elevation to the top of the obstacle. Suppose that the radar detects a tower that is 50,000 feet away, with an angle of elevation of 0.5 degrees. By how many feet must the plane rise in order to pass above the tower? ***Solution:*** The sketch below shows that there is a right triangle with a hypotenuse of 50,000 (ft) and smallest angle 0.5 (degrees). To find the side opposite this angle, which represents the minimum height the plane should rise, students would use: $\sin 0.5° = \frac{h}{50{,}000}$, so that $h = (50{,}000)\sin 0.5° \approx 436.33$ ft.	



Similarity, Proof, and Trigonometry

Apply trigonometry to general triangles.

M.GHS.22

Derive the formula A = 1/2 ab sin(C) for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

M.GHS.23

Prove the Laws of Sines and Cosines and use them to solve problems. Instructional Note: With respect to the general case of the Laws of Sines and Cosines, the definitions of sine and cosine must be extended to obtuse angles.

M.GHS.24

Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles. Instructional Note: With respect to the general case of the Laws of Sines and Cosines, the definitions of sine and cosine must be extended to obtuse angles.

Students advance their knowledge of right-triangle trigonometry by applying trigonometric ratios in non-right triangles. For instance, students see that by dropping an altitude in a given triangle, they divide the triangle into right triangles to which these relationships can be applied. By seeing that the base of the triangle is a and the height is $b \bullet \sin C$, students derive a general formula for the area of any triangle $A = \frac{1}{2}ab\sin C$ (**M.GHS.22**). In addition, they use reasoning about similarity and trigonometric identities to derive the Laws of Sines and Cosines only in acute triangles, and they use these and other relationships to solve problems (**M.GHS.23–M.GHS.24**). Instructors will need to address the ideas of the sine and cosine of angles larger than or equal to 90 degrees to fully discuss Laws of Sine and Cosine, although full unit-circle trigonometry need not be discussed in this course.

Circles With and Without Coordinates

Understand and apply theorems about circles.

M.GHS.34
Prove that all circles are similar.

M.GHS.35
Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

M.GHS.36
Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

M.GHS.37
Construct a tangent line from a point outside a given circle to the circle.

Students extend their understanding of the usefulness of similarity transformations by investigating circles (**M.GHS.34**). For instance, students can reason that any two circles are similar by describing precisely how to transform one into the other, as the following example illustrates with two specific circles. Students continue investigating properties of circles and relationships among angles, radii, and chords (**M.GHS.35–M.GHS.37**).

Example	**M.GHS.34**

Students can show that the two circles C and D given by the equations below are similar.

$$C: (x-1)^2 + (y-4)^2 = 9$$
$$D: (x+2)^2 + (y-1)^2 = 25$$

Solution:
Because the centers of the circles are $(1,4)$ and $(-2,1)$, respectively, the first step is to translate the center of circle C to the center of circle D using the translation $T(x,y) = (x-3, y-3)$. The final step is to dilate from the point $(-2,1)$ by a scale factor of $\frac{5}{3}$, since the radius of circle C is 3 and the radius of circle D is 5.

Circles With and Without Coordinates

Find arc lengths and areas of sectors of circles.

M.GHS.38

Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. Instructional Note: Emphasize the similarity of all circles. Reason that by similarity of sectors with the same central angle, arc lengths are proportional to the radius. Use this as a basis for introducing radian as a unit of measure. It is not intended that it be applied to the development of circular trigonometry in this course.

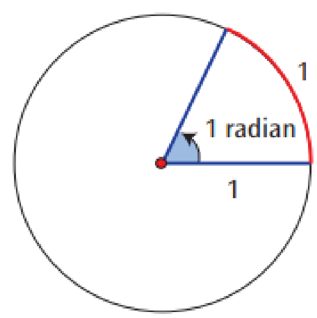
Another important application of the concept of similarity is the definition of the radian measure of an angle. Students can derive this result in the following way: given a sector of a circle C of radius r and central angle α, and a sector of a circle D of radius s and central angle also α, it stands to reason that because these sectors are similar,

$$\frac{\text{length of arc on circle } C}{r} = \frac{\text{length of arc on circle } D}{s}$$

Therefore, much like defining the trigonometric functions, there is a constant m such that for an arc subtended by an angle α on any circle:

$$\frac{\text{length of arc subtended by angle } \alpha}{\text{radius of the circle}} = m.$$

This constant of proportionality is the *radian measure* of angle α. It follows that an angle that subtends an arc on a circle that is the same length as the radius measures 1 radian. By investigating circles of different sizes, using string to measure arcs subtended by the same angle, and finding the ratios described above, students can apply their proportional-reasoning skills to discover this constant ratio, thereby developing an understanding of the definition of *radian measure*.



Connecting Algebra and Geometry Through Coordinates

Use coordinates to prove simple geometric theorems algebraically.

M.GHS.29

Use coordinates to prove simple geometric theorems algebraically. (e.g., Prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0, 2).

M.GHS.30
Prove the slope criteria for parallel and perpendicular lines and uses them to solve geometric problems. (e.g., Find the equation of a line parallel or perpendicular to a given line that passes through a given point.) Instructional Note: Relate work on parallel lines to work in High School Algebra I involving systems of equations having no solution or infinitely many solutions.

M.GHS.31
Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

M.GHS.32
Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. This standard provides practice with the distance formula and its connection with the Pythagorean theorem.

Translate between the geometric description and the equation for a conic section.
M.GHS.33
Derive the equation of a parabola given a focus and directrix. Instructional Note: The directrix should be parallel to a coordinate axis.

Standards **M.GHS.29** and **M.GHS.31** call for students to continue their work of using coordinates to prove geometric theorems with algebraic techniques. In standard **M.GHS.31**, given a directed line segment represented by a vector emanating from the origin to the point $(4,6)$, students may be asked to find the point on this vector that partitions it into a ratio of 2:1. Students may construct right triangles and use triangle similarity to find this point, or they may represent the vector as $x = 4t$, $y = 6t$ for $0 \leq t \leq 1$ and reason that the point they seek can be found when $t = \frac{2}{3}$.

Circles With and Without Coordinates

Translate between the geometric description and the equation for a conic section.
M.GHS.39
Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Use coordinates to prove simple geometric theorems algebraically.
M.GHS.40
Use coordinates to prove simple geometric theorems algebraically. (e.g., Prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point $(1, \sqrt{3})$ lies on the circle centered at the origin and containing the point $(0, 2)$.)

Instructional Note: Include simple proofs involving circles.

Apply geometric concepts in modeling situations.

M.GHS.41

Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). Instructional Note: Focus on situations in which the analysis of circles is required.

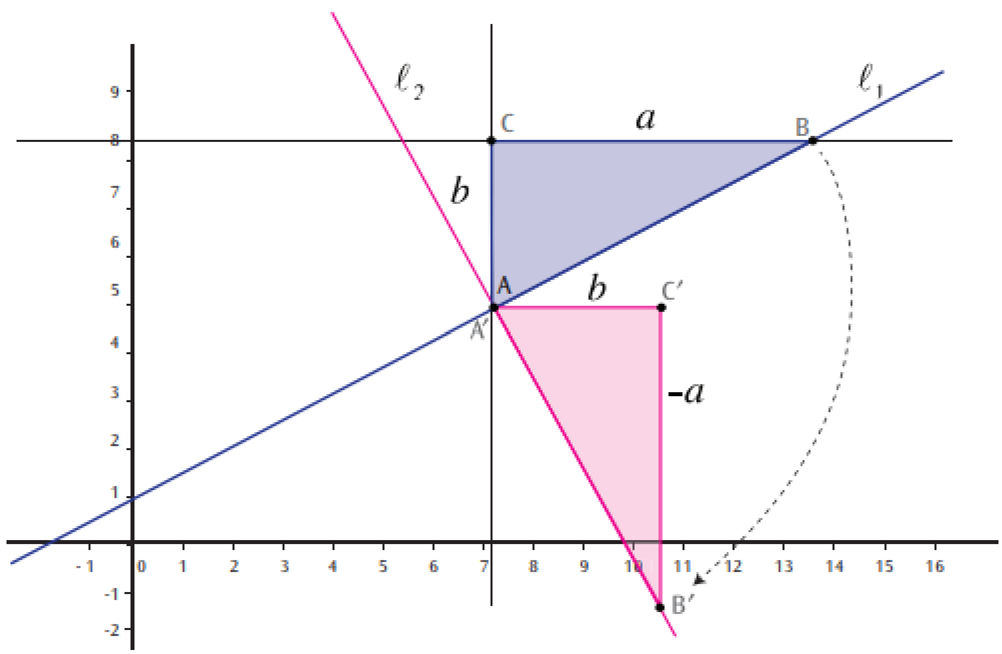
The largest intersection of algebra concepts and geometry occurs here, wherein two-dimensional shapes are represented on a coordinate system and can be described using algebraic equations and inequalities. A derivation of the equation of a circle by the Pythagorean Theorem and the definition of a circle (**M.GHS.39**) is as follows: given that a circle consists of all points (x, y) that are at a specific distance $r > 0$ from a fixed center (h, k), students see that $\sqrt{(x-h)^2 + (y-k)^2} = r$ for any point lying on the circle, so that $(x-h)^2 + (y-k)^2 = r^2$ determines this circle. Students can derive this equation and flexibly change an equation into this form by completing the square as necessary. By understanding the derivation of this equation, students develop a clear meaning of the variables h, k, and r. Standard **M.GHS.33** calls for students to do the same for the definition of a parabola in terms of a focus and directrix.

Many simple geometric theorems can be proved algebraically, but two results of high importance are the slope criteria for parallel and perpendicular lines. In grade seven, students begin to study lines and linear equations; in the Geometry course, they not only use relationships between slopes of parallel and perpendicular lines to solve problems, but they also justify why these relationships are true. An intuitive argument for why parallel lines have the same slope might read, "Since the two lines never meet, each line must *keep up* with the other as we travel along the slopes of the lines. So it seems obvious that their slopes must be equal." This intuitive thought leads to an equivalent statement: If given a pair of linear equations $l_1: y = m_1x + b_1$ and $l_2: y = m_2x + b_2$ (for $m_1, m_2 \neq 0$) such that $m_1 \neq m_2$— that is, such that *their slopes are different*— then the lines must intersect. Solving for the intersection of the two lines yields the x-coordinate of their intersection to be $x = \frac{b_2 - b_1}{m_2 - m_1}$, which surely exists because $m_1 \neq m_2$. It is important for students to understand the steps of the argument and comprehend why proving this statement is equivalent to proving the statement "If $l_1 \parallel l_2$, then $m_1 = m_2$" (**MHM1, MHM2**).

In addition, students are expected to justify why the slopes of two non-vertical perpendicular lines l_1 and l_2 satisfy the relationship $m_1 = -\frac{1}{m_2}$, or $m_1 \bullet m_2 = -1$. Although there are numerous ways to do this, only one way is presented here, and dynamic geometry software can be used to illustrate it well (**MHM4**). Let l_1 and l_2 be any two non-vertical perpendicular lines. Let A be the intersection of the two lines, and let B be any other point on l_1 above A. A vertical line is drawn through A, a horizontal line is drawn through B, and C is the intersection of those two lines.

ΔABC is a right triangle. If a is the horizontal displacement Δx from C to B, and b is the length of $\overline{AC}$, then the slope of l_1 is $m_1 = \frac{\Delta y}{\Delta x} = \frac{a}{b}$. By rotating ΔABC clockwise around A by 90 degrees, the hypotenuse $\overline{AB'}$ of the rotated triangle $\Delta AB'C'$ lies on l_2. Using the legs of $\Delta AB'C'$, students see that the slope of l_2 is $m_2 = \frac{\Delta y}{\Delta x} = \frac{-a}{b}$. Thus $m_1 \bullet m_2 = \frac{b}{a} \bullet \frac{-a}{b} = -1$. The following figure gives a visual presentation of this proof (**MHM1, MHM7**).

Illustration of the Proof That the Slopes of Two Perpendicular Lines are Opposite Reciprocals of One Another

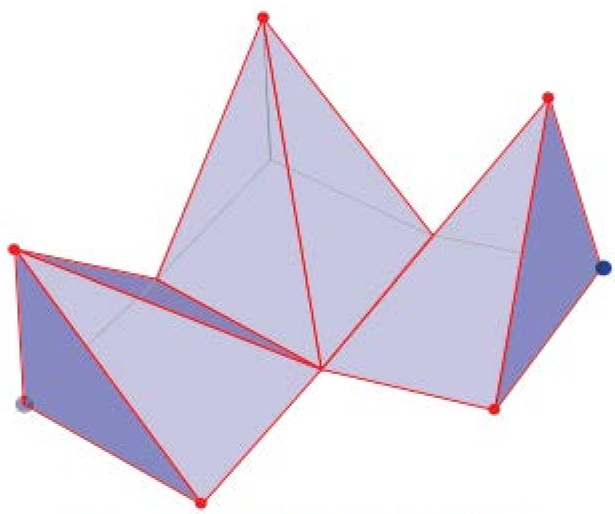


The proofs described above make use of several ideas that students learn in Geometry and prior courses—for example, the relationship between equations and their graphs in the plane (**M.A1HS.15**) and solving equations with variable coefficients (**M.A1HS.10**). An investigative approach that first uses several examples of lines that are perpendicular and their equations to find points, construct triangles, and decide if the triangles formed are right triangles will help students ramp up to the second proof (**MHM8**). Once more, the reasoning required to make sense of such a proof and to communicate the essence of the proof to a peer is an important goal of geometry instruction (**MHM3**).

Extending to Three Dimensions
Explain volume formulas and use them to solve problems. **M.GHS.25** Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments. Instructional Note: Informal arguments for area and volume formulas can make use of the way in which area and volume scale under similarity transformations: when one figure in the plane results from another by applying a similarity transformation with scale factor k, its area is k^2 times the area of the first. Similarly, volumes of solid figures scale by k^3 under a similarity transformation with scale factor k. **M.GHS.26** Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. Instructional Note: Informal arguments for area and volume formulas can make use of the way in which area and volume scale under similarity transformations: when one figure in the plane results from another by applying a similarity transformation with scale factor k, its area is k^2 times the area of the first. Similarly, volumes of solid figures scale by k^3 under a similarity transformation with scale factor k. **Visualize the relation between two dimensional and three-dimensional objects.** **M.GHS.27** Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects. **Apply geometric concepts in modeling situations.** **M.GHS.28** Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). Instructional Note: Focus on situations that require relating two- and three-dimensional objects, determining and using volume, and the trigonometry of general triangles.

The ability to visualize two- and three-dimensional shapes is a useful skill. This group of standards addresses that skill and includes understanding and using volume and area formulas for curved objects. Students also have the opportunity to make use of the notion of a *limiting process*—an idea that plays a large role in calculus and advanced mathematics courses—when they investigate the formula for the area of a circle. By experimenting with grids of finer and finer mesh, students can repeatedly approximate the area of a unit circle and thereby get a better and better approximation for the irrational number π. They also dissect shapes and make arguments based on these dissections. For instance, as shown in the following figure, a cube can be dissected into three congruent pyramids, which can lend weight to the formula that the volume of a pyramid of base area B and height h is $\frac{1}{3}Bh$ (**MHM2**).

Three Congruent Pyramids That Form a Cube



Source: Park City Mathematics Institute 2013.

Modeling with Geometry

Visualize relationships between two dimensional and three-dimensional objects and apply geometric concepts in modeling situations.

M.GHS.53
Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

M.GHS.54
Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).

M.GHS.55
Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

This set of standards is rich with opportunities for students to apply modeling (**MHM4**) with geometric concepts. The implementation of these standards should not be limited to the end of a Geometry course simply because they are later in the sequence of standards; they should be employed throughout the geometry curriculum. In standard **M.GHS.53**, students use geometric shapes, their measures, and their properties to describe objects. This standard can involve two- and three-dimensional shapes, and it is not relegated to simple applications of formulas. In standard **M.GHS.55**, students solve design problems by modeling with geometry, such as the one illustrated below.

Example: Ice-Cream Cone **M.GHS.55**

The owner of a local ice-cream parlor has hired you to assist with his company's new venture: the company will soon sell its ice-cream cones in the freezer section of local grocery stores. The manufacturing process requires that each ice-cream cone be wrapped in a cone-shaped paper wrapper with a flat, circular disc covering the top. The company wants to minimize the amount of paper that is wasted in the process of wrapping the cones. Use a real ice-cream cone or the dimensions of a real ice-cream cone to complete the following tasks:

a. Sketch a wrapper like the one described above, using the actual size of your cone. Ignore any overlap required for assembly.
b. Use your sketch to help develop an equation the owner can use to calculate the surface area of a wrapper (including the lid) for another cone, given that its base had a radius of length r and a slant height s.
c. Using measurements of the radius of the base and slant height of your cone, and your equation from step b, find the surface area of your cone.
d. The company has a large rectangular piece of paper that measures 100 centimeters by 150 centimeters. Estimate the maximum number of complete wrappers sized to fit your cone that could be cut from this single piece of paper, and explain your estimate. (Solutions can be found at https://www.illustrativemathematics.org/ [accessed April 1, 2015].)

Source: Illustrative Mathematics 2013l.

Conceptual Category: Statistics and Probability

In grades seven and eight, students learn some basic concepts related to probability, including chance processes, probability models, and sample spaces. In higher mathematics, the relative-frequency approach to probability is extended to conditional probability and independence, rules of probability and their use in finding probabilities of compound events, and the use of probability distributions to solve problems involving expected value (University of Arizona [UA] Progressions Documents, 2012d). Building on probability concepts that begin in the middle grades, students in the Geometry course use the language of set theory to expand their ability to compute and interpret theoretical and experimental probabilities for compound events, attending to mutually exclusive events, independent events, and conditional probability. Students should make use of geometric probability models wherever possible and use probability to make informed decisions (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010a).

Applications of Probability

Understand independence and conditional probability and use them to interpret data.

M.GHS.42
Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events (“or,” “and,” “not”).

M.GHS.43
Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.

M.GHS.44
Recognize the conditional probability of A given B as P(A and B)/P(B), and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B. Instructional Note: Build on work with two-way tables from Algebra I to develop understanding of conditional probability and independence.

M.GHS.45
Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results. Instructional Note: Build on work with two-way tables from Algebra I to develop understanding of conditional probability and independence.

M.GHS.46
Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.

To develop student understanding of conditional probability, students should experience two types of problems: those in which the uniform probabilities attached to outcomes lead to independence of the outcomes, and those in which they do not (**M.GHS.42–M.GHS.44**). The following examples illustrate these two distinct possibilities.

Example: Guessing on a True–False Quiz	**M.GHS.42–44**

If there are four true-or-false questions on a quiz, then the possible outcomes based on guessing on each question may be arranged as in the table below:

Possible outcomes: Guessing on four true-false questions

Number correct	Outcomes	Number correct	Outcomes	Number correct	Outcomes
4	CCCC	2	CCII	1	CIII
3	ICCC	2	CICI	1	ICII
3	CICC	2	CIIC	1	IICI
3	CCIC	2	ICCI	1	IIIC
3	CCCI	2	ICIC	0	IIII
		2	IICC		

C indicates a correct answer; I indicates an incorrect answer.

By counting outcomes, one can find various probabilities. For example:

$$P(C \text{ on the first question}) = \frac{1}{2}$$

and

$$P(C \text{ on second question}) = \frac{1}{2}$$

Noticing that P[(C on first) AND (C second)] $= \frac{4}{16} = \frac{1}{2} \bullet \frac{1}{2}$ shows that the two events—getting the first question correct and the second question correct—are independent.

Adapted from UA Progressions Documents 2012d.

Example: Work-Group Leaders	**M.GHS.42–44**

Suppose a five-person work group consisting of three girls (April, Briana, and Cyndi) and two boys (Daniel and Ernesto) wants to randomly choose two people to lead the group. The first person is the discussion leader and the second is the recorder, so order is important in selecting the leadership team. In the table below, “A” represents April, “B” represents Briana, “C” represents Cyndi, “D” represents Daniel, and “E” represents Ernesto. There are 20 outcomes for this situation:

Selecting two students from three girls and two boys		
Number of girls	Outcomes	
2	AB	BA
2	AC	CA
2	BC	CB
1	AD	DA
1	AE	EA
1	BD	DB
1	BE	EB
1	CD	DC
1	CE	EC
0	DE	ED

Notice that the probability of selecting two girls as the leaders is as follows:

$$P(\text{two girls chosen}) = \frac{6}{20} = \frac{3}{10}$$

whereas

$$P(\text{girl selected on first draw}) = \frac{12}{20} = \frac{3}{5}$$

and

$$P(\text{girl selected on second draw}) = \frac{3}{5}$$

But since $\frac{3}{5} \bullet \frac{3}{5} \neq \frac{3}{10}$, the two events are not independent.

One can also use the conditional-probability perspective to show that these events are not independent.

$$\text{Since } P(\text{girl on second} \mid \text{girl on first}) = \frac{6}{12} = \frac{1}{2}$$

and

$$P(\text{girl selected on second}) = \frac{3}{5},$$

these events are seen to be dependent.

Adapted from UA Progressions Documents 2012d.

Applications of Probability

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

M.GHS.47

Find the conditional probability of A given B as the fraction of B's outcomes that also belong to

A, and interpret the answer in terms of the model.

M.GHS.48
Apply the Addition Rule, P(A or B) = P(A) + P(B) – P(A and B), and interpret the answer in terms of the model.

M.GHS.49
Apply the general Multiplication Rule in a uniform probability model, P(A and B) = P(A)P(B|A) = P(B)P(A|B), and interpret the answer in terms of the model.

M.GHS.50
Use permutations and combinations to compute probabilities of compound events and solve problems.

Students explore finding probabilities of compound events (**M.GHS.47–M.GHS.50**) A simple experiment in which students roll two number cubes and tabulate the possible outcomes can shed light on these formulas before they are extended to application problems.

Example	**M.GHS.47–M.GHS.50**

On April 15, 1912, the RMS *Titanic* rapidly sank in the Atlantic Ocean after hitting an iceberg. Only 710 of the ship's 2,204 passengers and crew members survived. Some believe that the rescue procedures favored the wealthier first-class passengers. Data on survival of passengers are summarized in the table at the end of this example, and these data will be used to investigate the validity of such claims. Students can use the fact that two events A and B are independent if $P(A|B) = P(A) \bullet P(B)$. A represents the event that a passenger survived, and B represents the event that the passenger was in first class. The conditional probability $P(A|B)$ is compared with the probability $P(A)$.

For a first-class passenger, the probability of surviving is the fraction of all first-class passengers who survived. That is, the sample space is restricted to include only first-class passengers to obtain:

$$P(A|B) = \frac{202}{325} \approx 0.622$$

The probability that a passenger survived is the number of all passengers who survived divided by the total number of passengers:

$$P(A) = \frac{498}{1316} \approx 0.378$$

Since $0.622 \neq 0.378$, the two given events are not independent. Moreover, it can be said that being a passenger in first class did increase the chances of surviving the accident.

Students can be challenged to further investigate where similar reasoning would apply today. For example, what are similar statistics for Hurricane Katrina, and what would a similar analysis conclude about the distribution of damages? (**MHM4**)

***Titanic* passengers**	**Survived**	**Did not survive**	**Total**
First-class	202	123	325
Second-class	118	167	285
Third-class	178	528	706
Total passengers	498	818	1316

Adapted from Illustrative Mathematics 2013q.

Applications of Probability

Use probability to evaluate outcomes of decisions.

Instructional Note: This unit sets the stage for work in Algebra II, where the ideas of statistical inference are introduced. Evaluating the risks associated with conclusions drawn from sample data (i.e. incomplete information) requires an understanding of probability concepts.

M.GHS.51
Use probabilities to make fair decisions (e.g., drawing by lots and/or using a random number generator).

M.GHS.52
Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, and/or pulling a hockey goalie at the end of a game).

Standards **M.GHS.51** and **M.GHS.52** involve students' use of probability models and probability experiments to make decisions. These standards set the stage for more advanced work in Algebra II, where the ideas of statistical inference are introduced. See the University of Arizona Progressions document titled "High School Statistics and Probability" for further explanation and examples: http://ime.math.arizona.edu/progressions/ (UA Progressions Documents 2012d [accessed April 6, 2015]).

Common Misconceptions – By Domain

Number and Quantity

- Students often have difficulty differentiating between finding the solution to an equation such as $x^2 = 25$ and taking the square root of a number. The equation $x^2 = 25$ has two solutions, 5 and -5. Expressions such as $\sqrt{3^2}$ or $\sqrt{(-7)^2}$ ask for the "principal" square root. As a result, $\sqrt{3^2}$ = 3 and $\sqrt{(-7)^2}$ = 7

- Student confusion in differentiating between expressions involving the square of a negative number and the negative of a square, often extend to square roots. Students may need help in appreciating the Order of Operations in differentiating between the expressions.

$$(-5)^2 \rightarrow (-5)(-5) \rightarrow 25$$

$$-(5)^2 \rightarrow -(5)(5) \rightarrow -25$$

$$\sqrt{(-5)^2} \rightarrow \sqrt{25} \rightarrow 5$$

$$-\sqrt{5^2} \rightarrow -\sqrt{25} \rightarrow -5$$

$$\sqrt{(5)^2} \rightarrow \sqrt{-25} \rightarrow 5$$

- Students may have difficulty when negative exponents involve fractions. Students may incorrectly believe that $16^{-\frac{1}{2}}$ means -4 rather than $\frac{1}{4}$.

$$16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$$

Algebra

- Students may have difficulty distinguishing between the terms of an expression and the variable of an expression. For example, students may reason that the trinomial $3a + b + 5cd$ has four terms because the student is counting variable rather than terms. Often students do not recognize a constant as a term and incorrectly identify the trinomial $4x - 2y + 7$ as having two terms.

- Students who rely solely on procedures may believe they need multiply the binomials in an equation such as $(x + 4)(x - 3) = 0$ and then factor the resulting expression to solve for the zeros. Students need to develop an understanding of the concept of the Zero

Factor Property. Students may experience similar confusion in applying the Zero Factor Property in solving equations such as $5(x-7)(x+1)=0$ or $6x(x-7)=0$.

- Students who mistakenly believe that $\sqrt{x^2+\ y^2}$ is equivalent to $x+y$, may deduce this from the misconception that $(x+y)^2$ is equivalent to $x^2+\ y^2$.

- Students who mistakenly equate $(x+y)^2$ with $x^2+\ y^2$ or equate $(x-y)^2$ with x^2-y^2 have misconceptions regarding the concept of finding the square of a number or expression. These misconceptions may stem from students' prior difficulty in recognizing that $(-5)^2$ → (-5)(-5) →25 or that $-(5)^2$ → –(5)(5) → -25.

- Students who incorrectly equate rational expressions such as $\frac{x^2-6x+9}{x-3}$ with $x-2x+9$ or $x-6x-3$ or x^2+2x+9, etc., have difficulty distinguishing between terms and factors.

- Students may demonstrate difficulties in solving the equation $\sqrt{x-1}$ = $x-7$. Students may appropriately decide to square both sides of the equation and write $(\sqrt{x-1}\,)^2$ = $(x-7)^2$ Student error may arise in writing $(\sqrt{x-1}\,)^2$ = x^2 - 7^2, demonstrating misconceptions in squaring binomials (or multiplying) binomials.

- The procedure for solving equations and inequalities are so similar. As a result, students often forget to attend to precision when multiplying and dividing by a negative number when solving inequalities. Students may correctly solve equations such as $-2x=6$ and find its solution to be $x=-3$. Misconceptions arise when students equate solving the equation with a similar inequality $-2x\ \leq 6$ and incorrectly determine that $x\ \leq -3$. The misconception can be addressed by encouraging students to verify solutions to equations and to inequalities.

- Students often solve rational and radical equations without checking to determine if any solution may be erroneous. Students should be encouraged to verify solutions.

Functions

- Students may erroneously interpret the notation $g(3)$ to mean "$g\ times\ 3$".

- Students often believe that all functions must use the symbols f, x, and y.

- When graphing students may confuse the parts of the slope-equation form of a linear equation. Students may incorrectly determine that the function $g(x) = 2x + \frac{3}{5}$ has a slope of $\frac{3}{5}$ and a y-intercept of 2. Students may be relying on a procedural understanding that the y-intercept is always an integer and the slope or $\frac{rise}{run}$ must be a fraction.

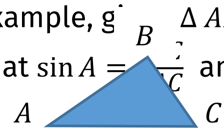
- Students often incorrectly assume that the function $f(x + k)$ where $k > 0$ will result in a horizontal shift of the graph k units to the right.

Geometry

- Students often think of congruence as "figures with the same shape and size." While this understanding is not incorrect, it is important to that the continually emphasize the link of congruence with rigid motions and show that rigid motions do in fact produce "figures with the same shape and size."

- Students may look at scale factor as the distance that is added on to the original distance. Dilations where the center of dilation is a vertex of a figure can prove challenging because the sides of the pre-image and image overlap.

- Students may have difficulty identifying the relevant sides in relationship to a given angle, especially If the triangle is not depicted in a typical position where its legs are horizontal and vertical. For example, g[illegible] ΔABC below with right angle B, students may struggle to understand that $\sin A =$ [illegible] and that $\sin A \neq \frac{BC}{AB}$.



- Students may have difficulty differentiating between the meaning of the statements $\Delta ABC \cong \Delta DEF$ and $\Delta ABC \sim \Delta DEF$. While $\Delta ABC \cong \Delta DEF$ implies that AB = DE, $\Delta ABC \sim \Delta DEF$ does not imply that AB = DE.

- Students often have difficulty differentiating among $\overleftarrow{EF}$, $\overrightarrow{EF}$, $\overleftrightarrow{EF}$, and $\overline{EF}$.

Statistics and Probability

- Students often use the word *outlier* inaccurately, failing to verify that it satisfies the necessary conditions. Using terms such as *unusual feature* or *data point* can help students avoid using the term outlier when it is not appropriate.

- Students commonly believe that any data that is collected should follow a normal distribution.

- Students need to understand that association does not necessarily provide evidence for cause and effect.

Steven L. Paine, Ed.D.
West Virginia Superintendent of Schools