West Virginia math4life

Educators' Guide for Mathematics

Mathematics II

West Virginia DEPARTMENT OF EDUCATION

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Mathematics II

The Mathematics II course focuses on quadratic expressions, equations, and functions and on comparing the characteristics and behavior of these expressions, equations, and functions to those of linear and exponential relationships from Mathematics I. The need for extending the set of rational numbers arises, and students are introduced to real and complex numbers. Links between probability and data are explored through conditional probability and counting methods and involve the use of probability and data in making and evaluating decisions. The study of similarity leads to an understanding of right-triangle trigonometry and connects to quadratics through Pythagorean relationships. Circles, with their quadratic algebraic representations, are addressed in the course. The courses in the Integrated Pathway follow the structure introduced in the K–8 grade levels of the West Virginia College- and Career- Readiness Standards; they present mathematics as a coherent subject and blend standards from different conceptual categories.

The standards in the integrated Mathematics II course come from the following conceptual categories: modeling, functions, number and quantity, algebra, geometry, and statistics and probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Mathematics II

In Mathematics II, students extend the laws of exponents to rational exponents and explore distinctions between rational and irrational numbers by considering their decimal representations. Students learn that when quadratic equations do not have real solutions, the number system can be extended so that solutions exist, analogous to the way in which extending whole numbers to negative numbers allows $x + 1 = 0$ to have a solution. Students explore relationships between number systems: whole numbers, integers, rational numbers, real numbers, and complex numbers. The guiding principle is that equations with no solutions in one number system may have solutions in a larger number system.

Students consider quadratic functions, comparing the key characteristics of quadratic functions to those of linear and exponential functions. They select from these functions to model phenomena. Students learn to anticipate the graph of a quadratic function by interpreting various forms of quadratic expressions. In particular, they identify the real solutions of a quadratic equation as the zeros of a related quadratic function. Students also learn that when quadratic equations do not have real solutions, the graph of the related quadratic function does not cross the horizontal axis. Additionally, students expand their experience with functions to include more specialized functions—absolute value, step, and other piecewise-defined functions.

Students in Mathematics II focus on the structure of expressions, writing equivalent expressions to clarify and reveal aspects of the quantities represented. Students create and solve equations, inequalities, and systems of equations involving exponential and quadratic expressions.

Building on probability concepts introduced in the middle grades, students use the language of set theory to expand their ability to compute and interpret theoretical and experimental probabilities for compound events, attending to mutually exclusive events, independent events, and conditional probability. Students use probability to make informed decisions, and they should make use of geometric probability models whenever possible.

Students apply their earlier experience with dilations and proportional reasoning to build a formal understanding of similarity. They identify criteria for similarity of triangles, use similarity to solve problems, and apply similarity in right triangles to understand right-triangle trigonometry, with particular attention to special right triangles and the Pythagorean Theorem. In Mathematics II, students develop facility with geometric proof. They use what they know about congruence and similarity to prove theorems involving lines, angles, triangles, and other polygons. They also explore a variety of formats for writing proofs.

In Mathematics II, students prove basic theorems about circles, chords, secants, tangents, and angle measures. In the Cartesian coordinate system, students use the distance formula to write the equation of a circle when given the radius and the coordinates of its center, and the equation of a parabola with a vertical axis when given an equation of its horizontal directrix and the coordinates of its focus. Given an equation of a circle, students draw the graph in the coordinate plane and apply techniques for solving quadratic equations to determine intersections between lines and circles, between lines and parabolas, and between two circles. Students develop informal arguments to justify common formulas for circumference, area, and volume of geometric objects, especially those related to circles.

Examples of Key Advances from Mathematics I

Students extend their previous work with linear and exponential expressions, equations, and systems of equations and inequalities to quadratic relationships.

- A parallel extension occurs from linear and exponential functions to quadratic functions: students begin to analyze functions in terms of transformations.
- Building on their work with transformations, students produce increasingly formal arguments about geometric relationships, particularly around notions of similarity.

Connecting Mathematical Habits of Mind and Content

The Mathematical Habits of Mind (**MHM**) apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, relevant, and meaningful subject. The Mathematical Habits of Mind represent a picture of what it looks like for students to do mathematics and, to the extent possible, content instruction should include attention to appropriate practice standards.

The West Virginia College- and Career-Readiness Standards call for an intense focus on the most critical material, allowing depth in learning, which is carried out through the Mathematical Habits of Mind. Connecting content and practices happens in the context of *working on problems*, as is evident in the first Mathematical Habit of Mind ("Make sense of problems and persevere in solving them"). The following table offers examples of how students can engage in each Mathematical Habit of Mind in the Mathematics II course.

Mathematical Habits of Mind—Explanation & Examples for Mathematics II

Mathematical Habits of Mind	Explanation and Examples
MHM1 Make sense of problems and persevere in solving them.	Students persevere when attempting to understand the differences between quadratic functions and linear and exponential functions studied previously. They create diagrams of geometric problems to help make sense of the problems.
MHM2 Reason abstractly and quantitatively.	Quantitative reasoning entails the following habits: creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.
MHM3 Construct viable arguments and critique the reasoning of others.	Students construct proofs of geometric theorems based on congruence criteria of triangles. They understand and explain the definition of *radian measure*.
MHM4 Model with mathematics.	Students apply their mathematical understanding of quadratic functions to real-world problems. Students also discover mathematics through experimentation and examination of patterns in data from real-world contexts.
MHM5 Use appropriate tools strategically.	Students develop a general understanding of the graph of an equation or function as a representation of that object and use tools such as graphing calculators or graphing software to create graphs in more complex examples to interpret the results.
MHM6 Attend to precision.	Students begin to understand that a *rational number* has a specific definition and that *irrational numbers* exist. When deciding if an equation can describe a function, students make use of the definition of *function* by asking, "Does every input value have exactly one output value?"
MHM7 Look for and make use of structure.	Students apply the distributive property to develop formulas such as $(a \pm b)^2 = a^2 \pm 2ab + b^2$. They see that the expression $5 + (x - 2)^2$ takes the form of "5 plus 'something' squared," and therefore that expression can be no smaller than 5.

MHM8 Look for and express regularity in repeated reasoning.	Students notice that consecutive numbers in the sequence of squares 1, 4, 9, 16, and 25 always differ by an odd number. They use polynomials to represent this interesting finding by expressing it as $(n+1)^2 - n^2 = 2n+1$.

MHM4 holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. Modeling in higher mathematics centers on problems that arise in everyday life, society, and the workplace. Such problems may draw upon mathematical content knowledge and skills articulated in the standards prior to or during the Mathematics II course. Examples where specific Mathematical Habits of Mind can be implemented are noted in parentheses, with the mathematical standard(s) also listed.

Mathematics II Content Standards, by Conceptual Category

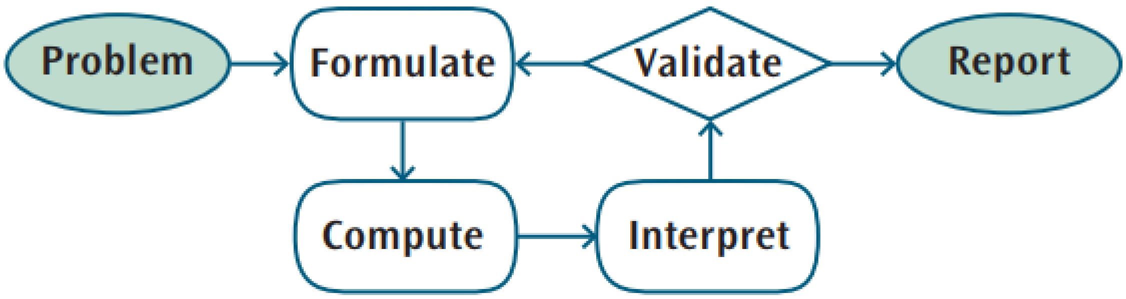
The Mathematics II course is organized by domains, clusters, and then standards. The overall purpose and progression of the standards included in Mathematics II are described below, according to each conceptual category.

Conceptual Category: Modeling

Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them and then constructing the mathematics in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise (e.g., Which of the quantities present in this situation are known, and which are unknown? Can a table of data be made? Is there a functional relationship in this situation?). Students need to decide on a solution path that may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They use previously derived models (e.g., linear functions), as well as new formulas or functions that apply. In addition, students may discover that answering their question requires solving an equation and knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, or the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see the modeling cycle presented below. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Modeling Cycle



The examples are framed as much as possible to illustrate the concept of mathematical modeling. The important ideas surrounding quadratic functions, graphing, solving equations, and rates of change are explored through this lens.

Conceptual Category: Functions

The standards of the functions conceptual category can serve as motivation for the study of standards in the other Mathematics II conceptual categories. Students have already worked with equations in which they have to "solve for x" as a search for the input of a function f that gives a specified output; solving the equation amounts to undoing the work of the function. The types of functions that students encounter in Mathematics II have new properties. For example, while linear functions show constant additive change and exponential functions show constant multiplicative change, quadratic functions exhibit a different change and can be used to model new situations. New techniques for solving equations need to be constructed carefully, as extraneous solutions may arise or no real-number solutions may exist. In general, functions describe how two quantities are related in a precise way and can be used to make predictions and generalizations, keeping true to the emphasis on modeling that occurs in higher mathematics. The core question when students investigate functions is, "Does each element of the domain correspond to exactly one element of the range?" (University of Arizona [UA] Progressions Documents, 2013c, 8).

Quadratic Functions and Modeling

Interpret functions that arise in applications in terms of a context.

M.2HS.7

For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. Instructional Note: Focus on quadratic functions; compare with linear and exponential functions studied in Mathematics I.

M.2HS.8

Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. (e.g., If the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.) Instructional Note: Focus on quadratic functions; compare with linear and exponential functions studied in Mathematics I.

M.2HS.9

Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. Instructional Note: Focus on quadratic functions; compare with linear and exponential functions studied in Mathematics I.

Analyze functions using different representations.

M.2HS.10

Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.
b. Graph square root, cube root and piecewise-defined functions, including step functions and absolute value functions. Instructional Note: Compare and contrast absolute value, step and piecewise defined functions with linear, quadratic, and exponential functions. Highlight issues of domain, range and usefulness when examining piecewise-defined functions.

Instructional Note: Extend work with quadratics to include the relationship between coefficients and roots and that once roots are known, a quadratic equation can be factored.

M.2HS.11
Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values and symmetry of the graph and interpret these in terms of a context.
b. Use the properties of exponents to interpret expressions for exponential functions. (e.g., Identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, and classify them as representing exponential growth or decay.) Instructional Note: This unit and, in particular, this standard extends the work begun in Mathematics I on exponential functions with integer exponents.

Instructional Note: Extend work with quadratics to include the relationship between coefficients and roots and that once roots are known, a quadratic equation can be factored.

M.2HS.12
Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). (e.g., Given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum). Instructional Note: Focus on expanding the types of functions considered to include, linear, exponential and quadratic. Extend work with quadratics to include the relationship between coefficients and roots and that once roots are known, a quadratic equation can be factored.

Standards **M.2HS.7– M.2HS.12** deal with understanding the concept of a function, interpreting characteristics of functions in context, and representing functions in different ways (**MHM6**). Standards **M.2HS.10– M.2HS.12** call for students to represent functions with graphs and identify key features of the graph. They represent the same function algebraically in different forms and interpret these differences in terms of the graph or context. For instance, students may easily see that the function $f(x) = 3x^2 + 9x + 6$ crosses the y-axis at (0,6), since the terms involving x are simply 0 when $x = 0$. But then they factor the expression defining f to obtain $f(x) = 3(x + 2)(x + 1)$, revealing that the function crosses the x-axis at (–2,0) and (–1,0) because those points correspond to where $f(x) = 0$ (**MHM7**). In Mathematics II, students work with linear, exponential, and quadratic functions and are expected to develop fluency with these types of functions, including the ability to graph them by hand.

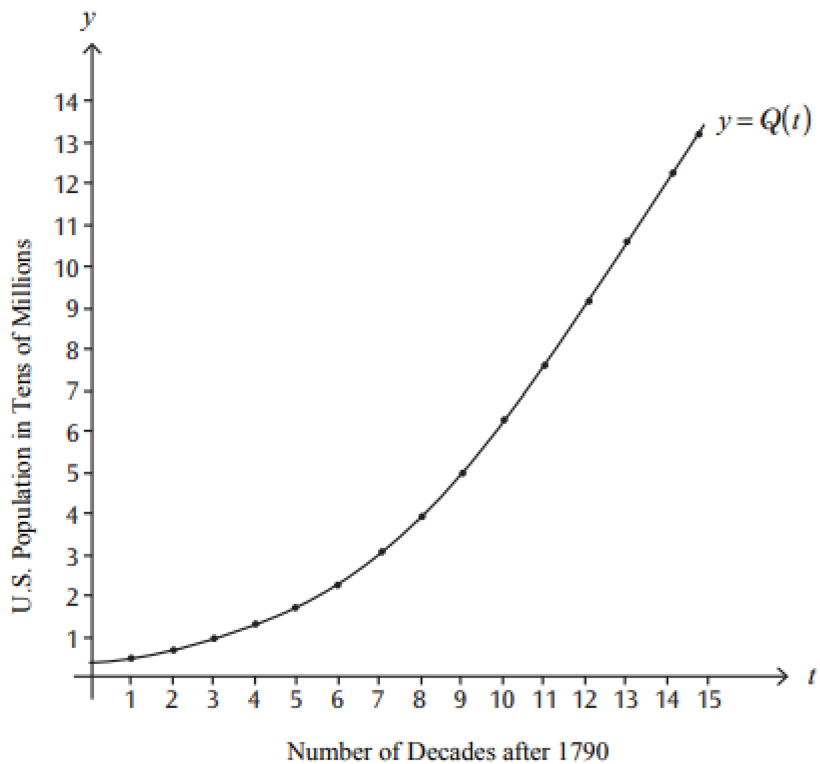
Students work with functions that model data and with choosing an appropriate model function by considering the context that produced the data. Students' ability to recognize rates of change, growth and decay, end behavior, roots, and other characteristics of functions becomes more sophisticated; they use this expanding repertoire of families of functions to inform their choices for models. **M.2HS.7– M.2HS.12** focus on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate.

Example: Population Growth **M.2HS.7– M.2HS.12**

The approximate population of the United States, measured each decade starting in 1790 through 1940, can be modeled with the following function:

$$P(t) = \frac{(3{,}900{,}000 \times 200{,}000{,}000)e^{0.31t}}{200{,}000{,}000 + 3{,}900{,}000(e^{0.31t} - 1)}$$

In this function, t represents the number of decades after 1790. Such models are important for planning infrastructure and the expansion of urban areas, and historically accurate long-term models have been difficult to derive.



Questions:

a. According to this model, what was the population of the United States in the year 1790?
b. According to this model, when did the U.S. population first reach 100,000,000? Explain your answer.

c. According to this model, what should the U.S. population be in the year 2010? Find the actual U.S. population in 2010 and compare with your result.

d. For larger values of t, such as $t = 50$, what does this model predict for the U.S. population? Explain your findings.

Solutions:

a. The population in 1790 is given by $P(0)$, which is easily found to be 3,900,000 because $e^{0.31(0)} = 1$.

b. This question asks students to find such that $P(t) = 100{,}000{,}000$. Dividing the numerator and denominator on the left by 100,000,000 and dividing both sides of the equation by 100,000,000 simplifies this equation to

$$\frac{3.9 \times e^{0.31t}}{200 + 3.9(e^{0.31t} - 1)} = 1.$$

Algebraic manipulation and solving for t result in $t \approx \frac{1}{031} \ln 50.28 \approx 12.64$. This means that after 1790, it would take approximately 126.4 years for the population to reach 100 million.

c. Twenty-two (22) decades after 1790, the population would be approximately 190,000,000, which is far less (by about 119,000,000) than the estimated U.S. population of 309,000,000 in 2010.

d. The structure of the expression reveals that for very large values of t, the denominator is dominated by $3{,}900{,}000e^{0.31t}$. Thus, for very large values of t,

$$P(t) \approx \frac{3{,}900{,}00 \times 200{,}000{,}000 \times e^{0.31t}}{3{,}900{,}000e^{0.31t}} = 200{,}000{,}000.$$

Therefore, the model predicts a population that stabilizes at 200,000,000 as increases.

Adapted from Illustrative Mathematics 2013m.

Quadratic Functions and Modeling

Build a function that models a relationship between two quantities.

M.2HS.13

Write a function that describes a relationship between two quantities.

a. Determine an explicit expression, a recursive process or steps for calculation from a context.

b. Combine standard function types using arithmetic operations. (e.g., Build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model. Instructional Note: Focus on situations that exhibit a quadratic or exponential relationship.

Build new functions from existing functions.

M.2HS.14

Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. Instructional Note: Focus on quadratic functions and consider including absolute value functions.

M.2HS.15

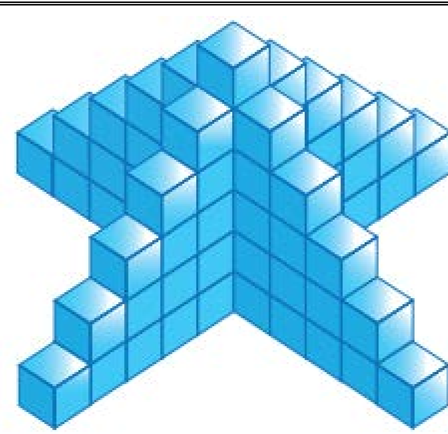
Find inverse functions. Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. For example, $f(x)= 2x^3$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$. Instructional Note: Focus on linear functions but consider simple situations where the domain of the function must be restricted in order for the inverse to exist, such as $f(x) = x^2$, $x > 0$.

Students in Mathematics II develop models for more complex or sophisticated situations than in previous courses because the types of functions available to them have expanded as in standard **M.2HS.13**. The following example illustrates the type of reasoning with functions that students are expected to develop in standard **M.2HS.13**.

Example: The Skeleton Tower **M.2HS.13**

The tower shown at right measures 6 cubes high.

a. How many cubes are needed to build this tower? (Organize your counting so other students can follow your reasoning.)
b. How many cubes would be needed to build a tower just like this one, but 12 cubes high? Justify your reasoning.
c. Find a way to calculate the number of cubes needed to build a similar tower that is n cubes high.



Solution:

a. The top layer has a single cube. The layer below has one cube beneath the top cube, plus 4 new ones, making a total of 5. The next layer has cubes below these 5, plus 4 new ones, to make 9. Continuing to add 4 each time gives a total of $1 + 5 + 9 + 13 + 17 + 21 = 66$ cubes in the skeleton tower with 6 layers.

b. Building upon the reasoning established in (a), the number of cubes in the bottom (12th) layer will be $1 + 4 \times 11$, since it is 11 layers below the top. So for this total, students need to add $1 + 5 + 9 + \ldots + 45$. One way to do this would be to add the numbers. Another method is the Gauss method: Rewrite the sum backward as $45 + 41 + 37 + \ldots + 1$. Now if this sum is placed below the previous sum, students can see that each pair of addends, one above the other, sums to 46. There are 12 columns, so the answer to this problem is half of $12 \times 46 = 552$, or 276.

c. Let $f(n)$ be the number of cubes in the nth layer counting down from the top. Then $f(1) = 1$, $f(2) = 5$, $f(3) = 9$, and so forth. In general, because each term is obtained from the previous one by adding 4, $f(n) = 4(n-1) + 1$. Therefore, the total for n layers in the tower is $1 + 5 + 9 + \cdots + f(n) = 1 + 5 + 9 + \cdots + (4(n-1) + 1)$. If the method from solution (b) is used here, twice this sum will be equal to $n \bullet (4(n-1) + 2)$, so the general solution for the number of cubes in a skeleton tower with n layers is $\frac{n(4(n-1)+2)}{2} = n(2n-1)$.

Note: For an alternative solution, visit https://www.illustrativemathematics.org/.

Adapted from Illustrative Mathematics 2013j

For standard **M.2HS.14**, students can make good use of graphing software to investigate the effects of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for different types of functions. For example, starting with the simple quadratic function $f(x) = x^2$, students see the relationship between the transformed functions $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ and the vertex form of a general quadratic, $f(x) = a(x - h)^2 + k$. They understand the notion of a *family of functions* and characterize such function families based on the properties of those families. In keeping with the theme of the input– output interpretation of a function, students should work toward developing an understanding of the effect on the output of a function under certain transformations, such as in the following table.

Expression	**Interpretation**
$f(a + 2)$	The output when the input is 2 greater than a
$f(a) + 3$	3 more than the output when the input is a
$2f(x) + 5$	5 more than twice the output of f when the input is x

Such understandings can help students see the effect of transformations on the graph of a function and, in particular, they can help students comprehend that the effect on the graph is the opposite to the transformation on the variable. For example, the graph of $y = f(x + 2)$ is the graph of f shifted 2 units to the left, not to the right (UA Progressions Documents 2013c, 7). These ideas are explored further with trigonometric functions in Mathematics III.

In standard **M.2HS.15**, students learn that some functions have the property that an input can be recovered from a given output—as with the equation $f(x) = c$, which can be solved for x given that c lies in the range of f. For example, a student might solve the equation $F = \frac{9}{5}C + 32$ for C. The student starts with this formula, showing how Fahrenheit temperature is a function of Celsius temperature, and by solving for C finds the formula for the inverse function. This is a contextually appropriate way to find the expression for an inverse function, in contrast with the practice of simply swapping x and y in an equation and solving for y.

Quadratic Functions and Modeling
Construct and compare linear, quadratic, and exponential models and solve problems.
M.2HS.16 Using graphs and tables, observe that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically; or (more generally) as a polynomial function. Instructional Note: Compare linear and exponential growth studied in Mathematics I to quadratic growth.

In Mathematics II, students continue their investigation of exponential functions by comparing them with linear and quadratic functions, observing that exponential functions will always grow larger than any polynomial function. Students may experiment with quadratic functions and discover how these functions can represent real-world phenomena such as projectile motion. A simple activity that involves tossing a ball and making a video recording of its height as it rises and falls can reveal that the height, as a function of time, is approximately quadratic. Afterward, students can derive a quadratic expression that determines the height of the ball at time using a graphing calculator or other software, and they can compare the values of the function with their data.

Similarity, Right Triangle Trigonometry, and Proof
Prove and apply trigonometric identities.
M.2HS.51 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find sin(θ), cos (θ), or tan (θ), given sin (θ), cos (θ), or tan (θ), and the quadrant of the angle. Instructional Note: Limit θ to angles between 0 and 90 degrees. Connect with the Pythagorean theorem and

the distance formula. Extension of trigonometric functions to other angles through the unit circle is included in Mathematics III.

Standard **M.2HS.51** is closely linked with standards **M.2HS.48– M.2HS.50**, but it is included here as a property of the trigonometric functions sine, cosine, and tangent. Students use the Pythagorean identity to find the output of a trigonometric function at given angle θ when the output of another trigonometric function is known.

Conceptual Category: Number and Quantity

Extending the Number System

Extend the properties of exponents to rational exponents.

M.2HS.1

Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. (e.g., We define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.)

M.2HS.2

Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers.

M.2HS.3

Explain why sums and products of rational numbers are rational, that the sum of a rational number and an irrational number is irrational and that the product of a nonzero rational number and an irrational number is irrational. Instructional Note: Connect to physical situations, e.g., finding the perimeter of a square of area 2.

In grade eight, students encounter some examples of irrational numbers, such as π and $\sqrt{2}$ (or $\sqrt{p}$ for p as a non-square number). In Mathematics II, students extend this understanding beyond the fact that there are numbers that are not rational; they begin to understand that rational numbers form a closed system. Students have witnessed that, with each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—integers, rational numbers, and real numbers—the distributive law continues to hold, and the commutative and associative laws are still valid for both addition and multiplication. However, in Mathematics II, students go further along this path. For example, with standard **M.2HS.3**, students may explain that the sum or product of two rational numbers is rational by arguing that the sum of two fractions with integer numerator and denominator is also a fraction of the same type, showing that the rational numbers are *closed* under the operations of addition and multiplication (**MHM3**). Moreover, they argue that the sum of a rational and an irrational is irrational, and the product of a non-zero rational and an irrational is still irrational, showing that irrational numbers are truly an additional set of numbers that, along with rational numbers, form a larger *system*: real numbers (**MHM3, MHM7**).

Standard **M.2HS.1** calls for students to make meaning of the representation of radicals with rational exponents. Students were first introduced to exponents in grade six; by the time they reach Mathematics II, they should have an understanding of the basic properties of exponents—for example, that

$$x^n x^m = x^{n+m},\ (x^n)^m = x^{nm},\ \frac{x^n}{x^m} = x^{n-m},\ x^0 = 1 \text{ for } x \neq 0, \text{ and so forth.}$$

In fact, students may have justified certain properties of exponents by reasoning about other properties (**MHM3, MHM7**). For example, they may have justified why any non-zero number to the power 0 is equal to 1:

$$x^0 = x^{n-n} = \frac{x^n}{x^n} = 1, \text{ for } x \neq 0$$

Students in Mathematics II further their understanding of exponents by using these properties to explain the meaning of rational exponents. For example, properties of whole-number exponents suggest that $\left(5^{1/3}\right)^3$ should be the same as $5^{[(1/3)\bullet 3]} = 5^1 = 5$, so that $5^{1/3}$ should represent the cube root of 5. In addition, recalling that $(ab)^n = a^n \bullet b^n$ reveals that

$$\sqrt{20} = (4 \bullet 5)^{1/2} = 4^{1/2} \bullet 5^{1/2} = 2\sqrt{5}.$$

This shows that $\sqrt{20} = 2\sqrt{5}$. The intermediate steps of writing the square root as a rational exponent are not entirely necessary here, but the principle of how to work with radicals based on the properties of exponents is. Students extend such work with radicals and rational exponents to variable expressions as well in standard **M.2HS.5**; for example, they rewrite an expression such as $(a^2b^5)^{3/2}$ by using radicals in standard **M.2HS.2**.

Extending the Number System
Perform arithmetic operations with complex numbers.
M.2HS.4 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form a + bi with a and b real.
M.2HS.5

Use the relation $i^2 = -1$ and the commutative, associative and distributive properties to add, subtract and multiply complex numbers. Instructional Note: Limit to multiplications that involve i^2 as the highest power of i.

Expressions and Equations

Use complex numbers in polynomial identities and equations.

M.2HS.24

Solve quadratic equations with real coefficients that have complex solutions. Instructional Note: Limit to quadratics with real coefficients.

M.2HS.25(+)

Extend polynomial identities to the complex numbers. For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$. Instructional Note: Limit to quadratics with real coefficients.

M.2HS.26(+)

Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials. Instructional Note: Limit to quadratics with real coefficients.

Standards identified with (+) are included to increase coherence but are not necessarily expected to be addressed on high stakes assessments.

In Mathematics II, students work with examples of quadratic functions and solve quadratic equations, encountering situations in which a resulting equation does not have a solution that is a real number (e.g., $(x-2)^2 = -25$). Students expand their extension of the concept of *number* to include complex numbers, numbers of the form $a + bi$, where is a number with the property that $i^2 = -1$, so that such an equation can be solved. They begin to work with complex numbers, first by finding simple square roots of negative numbers: $\sqrt{-25} = \sqrt{25 \bullet (-1)} = \sqrt{25} \bullet \sqrt{-(1)} = 5i$ (**MHM7**). They also apply their understanding of properties of operations (the commutative, associative, and distributive properties) and exponents and radicals to solve equations such as those mentioned above:

$$(x-2)^2 = -25, \text{ which implies } |x-2| = 5i, \text{ or } x = 2 \pm 5i$$

Now equations such as these have solutions, and the extended number system forms yet another system that behaves according to certain rules and properties as illustrated by standards **M.2HS.4– M.2HS.5 and M.2HS.24– M.2HS.26**. Additionally, by exploring examples of polynomials that can be factored with real and complex roots, students develop an understanding of the Fundamental Theorem of Algebra; they can show that this theorem is true for a quadratic polynomial by applying the quadratic formula and understanding the relationship between roots of a quadratic equation and the linear factors of the quadratic expression (**MHM2**).

Conceptual Category: Algebra

Students begin their work with expressions and equations in grades six through eight and extend their work to more complex expressions in Mathematics I. In Mathematics II, students encounter quadratic expressions for the first time and learn a new set of strategies for working with these expressions. As in Mathematics I, the Algebra conceptual category is closely tied to the Functions conceptual category, linking the writing of equivalent expressions, solving equations, and graphing to concepts involving functions.

Expressions and Equations

Interpret the structure of expressions.

M.2HS.17

Interpret expressions that represent a quantity in terms of its context.

a. Interpret parts of an expression, such as terms, factors, and coefficients.
b. Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret P(1 + r)n as the product of P and a factor not depending on P.

Instructional Note: Focus on quadratic and exponential expressions. Exponents are extended from the integer exponents found in Mathematics I to rational exponents focusing on those that represent square or cube roots.

M.2HS.18

Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. Instructional Note: Focus on quadratic and exponential expressions.

Write expressions in equivalent forms to solve problems.

M.2HS.19

Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

a. Factor a quadratic expression to reveal the zeros of the function it defines.
b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
c. Use the properties of exponents to transform expressions for exponential functions. For example the expression 1.15t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.

Instructional Note: It is important to balance conceptual understanding and procedural fluency in work with equivalent expressions. For example, development of skill in factoring and completing the square goes hand-in-hand with understanding what different forms of a quadratic expression reveal.

In Mathematics II, students extend their work with expressions to include examples of more complicated expressions, such as those that involve multiple variables and exponents. Students use the distributive property to investigate equivalent forms of quadratic expressions; for example, they write

$$\begin{aligned}(x+y)(x-y) &= x(x-y)+y(x-y)\\ &= x^2-xy+xy-y^2\\ &= x^2-y^2.\end{aligned}$$

This yields a special case of a factorable quadratic: the difference of squares. Students factor second- degree polynomials and simple third-degree polynomials by making use of such special forms and by using factoring techniques based on properties of operations—for example, factoring by grouping, which arises from the distributive property (**M.2HS.18**). Note that the standards do not mention "simplification" because it is not always clear what the simplest form of an expression is, and even in cases where it is clear, it is not obvious that the simplest form is desirable for a given purpose. The standards emphasize purposeful transformation of expressions into equivalent forms that are suitable for the purpose at hand (UA Progressions Documents 2013b, 5), as the following example shows.

Example: Which Is the Simpler Form? **M.2HS.18**

A particularly rich mathematical investigation involves finding a general expression for the sum of the first n consecutive natural numbers:

$$S = 1+2+3+\dots+(n-1)+n.$$

A famous tale speaks of a young C. F. Gauss being able to add the first 100 natural numbers quickly in his head, wowing his classmates and teachers alike. One way to find this sum is to consider the "reverse" of the sum:

$$S = n+(n-1)+(n-2)+\dots+3+2+1.$$

Then the two expressions for S are added together:

$$2S = (n+1)+(n+1)+\dots+(n+1)+(n+1)+(n+1).$$

where there are n terms of the form $(n+1)$. Thus, $2S = n(n+1)$, so that $S = \frac{n(n+1)}{2}$. While students may be tempted to transform this expression into $\frac{1}{2}n^2 + \frac{1}{2}n$, they are obscuring the ease with which they can evaluate the first expression. Indeed, since n is a natural number, one of either n or $n+1$ is even, so evaluating $\frac{n(n+1)}{2}$, especially mentally, is often easier. In Gauss's case, $\frac{100(101)}{2} = 50(101) = 5050$.

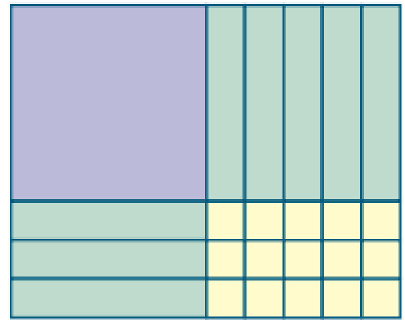
Students also use different forms of the same expression to reveal important characteristics of the expression. For instance, when working with quadratics, they complete the square in the expression $x^2 - 3x + 4$ to obtain the equivalent expression $\left(x - \frac{3}{2}\right)^2 + \frac{7}{4}$. Students can then reason with the new expression that the term being squared is always greater than or equal to 0; hence, the value of the expression will always be greater than or equal to $\frac{7}{4}$ in standard **M.2HS.19 (MHM3**). A spreadsheet or a computer algebra system may be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave, further contributing to students' understanding of work with expression (**MHM5**).

Extending the Number System
Perform arithmetic operations on polynomials.
M.2HS.6 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract and multiply polynomials. Instructional Note: Focus on polynomial expressions that simplify to forms that are linear or quadratic in a positive integer power of x.

To perform operations with polynomials meaningfully, students are encouraged to draw parallels between the set of integers and the set of all polynomials with real coefficients as addressed in standard **M.2HS.6 (MHM7**). Manipulatives such as "algebra tiles" may be used to support understanding of addition and subtraction of polynomials and the multiplication of monomials and binomials. Algebra tiles may be used to offer a concrete representation of the terms in a polynomial (**MHM5**). The tile representation relies on the area interpretation of multiplication: the notion that the product ab can be thought of as the area of a rectangle of dimensions a units and b units. With this understanding, tiles can be used to represent 1 square unit (a 1 by 1 tile), x square units (a 1 by x tile), and x^2 square units (an x by x tile). Finding the product $(x + 5)(x + 3)$ amounts to finding the area of an abstract rectangle of dimensions $(x + 5)$ and $(x + 3)$, as illustrated in the figure to the right (**MHM2**).

Care must be taken in the way negative numbers are handled with this representation, since, as with all models, there are potential limitations to connecting the mathematics to the representation. The tile representation of polynomials can support student understanding of the meaning of multiplication of variable expressions, and it is very useful for understanding the notion of completing the square (described later in this chapter).

Algebra Tiles



The rectangle above has height $(x+3)$ and base $(x+5)$. The total area represented, the product of these binomials, is seen to be $x^2 + 5x + 3x + 15 = x^2 + 8x + 15$.

Expressions and Equations

Create equations that describe numbers or relationships.

M.2HS.20
Create equations and inequalities in one variable and use them to solve problems. Instructional Note: Include equations arising from linear and quadratic functions, and simple rational and exponential functions. Extend work on linear and exponential equations in Mathematics I to quadratic equations.

M.2HS.21
Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. Instructional Note: Extend work on linear and exponential equations in Mathematics I to quadratic equations.

M.2HS.22
Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. (e.g., Rearrange Ohm's law V = IR to highlight resistance R.) Instructional Note: Extend to formulas involving squared variables. Extend work on linear and exponential equations in Mathematics I to quadratic equations.

In Mathematics II, students work with all available types of functions to create equations, including quadratic functions, absolute value functions, and simple rational and exponential functions (**M.2HS.20**). Although the functions used for standards **M.2HS.20– M.2HS.22** will often be linear, exponential, or quadratic, the types of problems should draw from more complex situations than those addressed in Mathematics I. Note that students are not required to study rational functions in Mathematics II.

To support **M.2HS.10** and **M.2HS.14**, it is suggested that absolute value be addressed. The basic absolute value function has at least two useful definitions: (1) a descriptive, verbal definition and (2) a formula definition. A common definition of the absolute value of x is

$|x|$ = the distance from the number x to 0 (on a number line).

An understanding of the number line easily yields that, for example, $|0| = 0$, $|7| = 7$, and $|-3.9| = 3.9$. However, an equally valid "formula" definition of absolute value reads as follows:

$$|x| = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$$

In other words, $|x|$ is simply x whenever is 0 or positive, but $|x|$ is the opposite of x whenever x is negative. Either definition can be extended to an understanding of the expression $|x - a|$ as the distance between x and a on a number line, an interpretation that has many uses. For a simple application of this idea, suppose a type of bolt is to be mass-produced in a factory with the specification that its width be 5 mm with an error no larger than 0.01 mm. If w represents the width of a given bolt produced on the production line, then w must satisfy the inequality $|w - 5| \leq 0.01$; that is, the difference between the actual width w and the target width should be less than or equal to 0.01 (**MHM4, MHM6**). Students should become comfortable with the basic properties of absolute values (e.g., $|x| + a \neq |x + a|$) and with solving absolute value equations and interpreting the solution.

In higher mathematics courses, intervals on the number line are often denoted by an inequality of the form $|x - a| \leq d$ for a positive number d. For example, $|x - 2| \leq \frac{1}{2}$ represents the closed interval $1\frac{1}{2} \leq x \leq 2\frac{1}{2}$. This can be seen by interpreting $|x - 2| = x - 2 \leq \frac{1}{2}$ as "the distance from x to 2 is less than or equal to $\frac{1}{2}$" and deciding which numbers fit this description.

On the other hand, in the case where x - 2 < 0, the following would hold true: $|x - 2| = -(x - 2) \leq \frac{1}{2}$, so that 2 . In the case where $x - s \geq 0$, $|x - 2| = x - 2 \leq \frac{1}{2}$, which means that $x \leq 2\frac{1}{2}$. Since students are looking for all values of x that satisfy both inequalities, the interval is $1\frac{1}{2} \leq x \leq 2\frac{1}{2}$. This shows how the formula definition can be used to find this interval.

Expressions and Equations

Solve equations and inequalities in one variable.

M.2HS.23

Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as a ± bi for real numbers a and b.

Instructional Note: Extend to solving any quadratic equation with real coefficients, including those with complex solutions.

Solve systems of equations.

M.2HS.27

Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. (e.g., Find the points of intersection between the line $y = –3x$ and the circle $x^2 + y^2 = 3$.) Instructional Note: Include systems that lead to work with fractions. (e.g., Finding the intersections between $x^2 + y^2 = 1$ and $y = (x+1)/2$ leads to the point (3/5, 4/5) on the unit circle, corresponding to the Pythagorean triple $3^2 + 4^2 = 5^2$.)

Students in Mathematics II extend their work with exponents to include quadratic functions and equations. To extend their understanding of these quadratic expressions and the functions defined by such expressions, students investigate properties of quadratics and their graphs in the Functions domain. It may be best to present the solving of quadratic equations in the context of functions. For instance, if the equation $h(t) = -16t^2 + 50t + 150$ defines the height of a projectile launched with an initial velocity of 50 ft/s from a height of 150 ft, then asking at which time the object hits the ground is asking for which t is found at $h(t) = 0$. That is, students now need to solve the equation $-16t^2 + 50t + 150 = 0$ and require new methods for doing so. Students investigate how to “undo” linear and simple exponential functions in Mathematics I; now they do so for quadratic functions and discover that the process is more complex.

Example	**M.2HS.23a**
When solving quadratic equations of the form $(x-p)^2 = q$, students rely on the understanding that they can take square roots of both sides of the equation to obtain the following: $$\sqrt{(x-p)^2} = \sqrt{q} \qquad (1)$$ In the case that $\sqrt{q}$ is a real number, this equation can be solved for x. A common mistake is to quickly introduce the $\pm$ symbol here, without understanding where the symbol comes from. Doing so without care often leads students to think that $\sqrt{9} = \pm 3$, for example. Note that the quantity $\sqrt{a^2}$ is simply a when $a \geq 0$ (as in $\sqrt{5^2} = \sqrt{25} = 5$), while $\sqrt{a^2}$ is equal to $-a$ (the opposite of a) when $a < 0$ (as in $\sqrt{(-4)^2} = \sqrt{16} = 4$). But this means that $\sqrt{a^2} = \|a\|$. Applying this to equation (1) yields $\|x-p\| = \sqrt{q}$. Solving this simple absolute value equation yields that $x - p = \sqrt{q}$ or $-(x-p) = \sqrt{q}$. This results in the two solutions $x = p + \sqrt{q}, p - \sqrt{q}$.	

Students also transform quadratic equations into the form $ax^2 + bx + c = 0$ for $a \neq 0$, which is the *standard form* of a quadratic equation. In some cases, the quadratic expression factors nicely and students can apply the zero product property of the real numbers to solve the resulting equation. The *zero product property* states that for two real numbers m and n, $m \bullet n = 0$ if and only if either $m = 0$ or $n = 0$. Hence, when a quadratic polynomial can be rewritten as $a(x-r)(x-s) = 0$, the solutions can be found by setting each of the linear factors equal to 0 separately, and obtaining the solution set $\{r, s\}$. In other cases, a means for solving a quadratic equation arises by *completing the square*. Assuming for simplicity that $a = 1$ in the standard equation above, and that the equation has been rewritten as $x^2 + bx = -c$, we can "complete the square" by adding the square of half the coefficient of the x-term to each side of the equation:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2 \qquad (2)$$

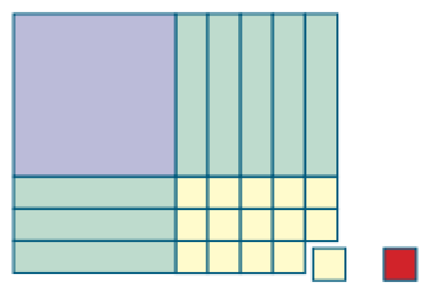
The result of this simple step is that the quadratic on the left side of the equation is a perfect square, as shown here:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Thus, we have now converted equation (2) into an equation of the form $(x-p)^2 = q$:

$$\left(x+\frac{b}{2}\right)^2=-c+\frac{b^2}{4}$$

This equation can be solved by the method described above, as long as the term on the right is non-negative. When $a \neq 1$, the case can be handled similarly and ultimately results in the familiar quadratic formula. Tile representations of quadratics illustrate that the process of completing the square has a geometric interpretation that explains the origin of the name. Students should be encouraged to explore these representations in order to make sense out of the process of completing the square (**MHM1, MHM5**).

Example: Completing the Square	**M.2HS.23a**
The method of completing the square is a useful skill in algebra. It is generally used to change a quadratic in standard form, ax^2+bx+c, into one in vertex form, $a(x-h)^2+k$. The vertex form can help determine several properties of quadratic functions. Completing the square also has applications in geometry standard **M.2HS.57** and later higher mathematics courses. To complete the square for the quadratic $y=x^2+8x+15$, half the coefficient of the x-term is squared to yield 16. Then students realize that they need to add 1 and subtract 1 to the quadratic expression: $$\begin{aligned} y &= x^2+8x+15+1-1 \\ &= x^2+8x+16-1 \end{aligned}$$ Factoring yields $y=(x+4)^2-1$.  In the picture at right, note that the tiles used to represent $x^2+8x+15$ have been rearranged to try to form a square, and that a positive unit tile and a negative unit tile are added into the picture to "complete the square."	

Conceptual Category: Geometry

In Mathematics I, students begin to formalize their understanding of geometry by defining congruence in terms of well-defined rigid motions of the plane. They find that congruence can be deduced in certain cases by investigating other relationships (e.g., that for triangles, the ASA, SAS, and SSS congruence criteria held). In Mathematics II, students further enrich their ability to reason deductively and begin to write more formal proofs of various geometric results. They also apply to triangles their knowledge of similarity and discover powerful relationships in right triangles, leading to the discovery of trigonometric functions. Finally, students' understanding of the Pythagorean relationship and their work with quadratics leads to algebraic representations of circles and more complex proofs of results in the plane.

Similarity, Right Triangle Trigonometry, and Proof

Prove geometric theorems.

M.2HS.42

Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints. Implementation may be extended to include concurrence of perpendicular bisectors and angle bisectors as preparation for M.2HS.C.3. Instructional Note: Encourage multiple ways of writing proofs, such as in narrative paragraphs, using flow diagrams, in two-column format, and using diagrams without words. Students should be encouraged to focus on the validity of the underlying reasoning while exploring a variety of formats for expressing that reasoning.

M.2HS.43

Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point. Instructional Note: Encourage multiple ways of writing proofs, such as in narrative paragraphs, using flow diagrams, in two-column format, and using diagrams without words. Students should be encouraged to focus on the validity of the underlying reasoning while exploring a variety of formats for expressing that reasoning. Implementation of this standard may be extended to include concurrence of perpendicular bisectors and angle bisectors in preparation for the unit on Circles With and Without Coordinates.

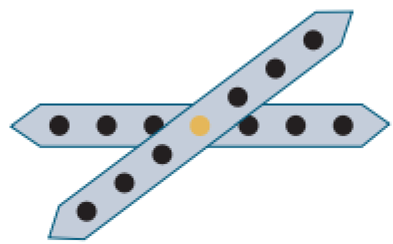
M.2HS.44

Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other and conversely, rectangles are parallelograms with congruent diagonals. Instructional Note: Encourage multiple ways of writing proofs, such as in narrative paragraphs, using flow diagrams, in two-column format and using diagrams without words. Students should be encouraged to focus on the validity of the underlying reasoning while exploring a variety of formats for expressing that reasoning.

Students prove the congruence criteria for triangles (ASA, SAS, and SSS) with the more basic notion of congruence by rigid motions. Instructors are encouraged to use a variety of strategies to engage students in understanding and writing proofs. Strategies include using numerous pictures to demonstrate results; using patty paper, transparencies, or dynamic geometry software to explore the relationships in a proof; creating flowcharts and other organizational diagrams for outlining a proof; and writing step-by-step or paragraph formats for a completed proof (**MHM5**). Above all else, instructors should emphasize the reasoning involved in connecting one step in the logical argument to the next. Students should be encouraged to make conjectures based on experimentation, to justify their conjectures, and to communicate their reasoning to their peers (**MHM3**). Such reasoning, justification, and communication in precise language are central to geometry instruction and should be emphasized.

Example: The Kite Factory	**M.2HS.44**

Kite engineers want to know how the shape of a kite—the length of the rods, where they are attached, the angle at which the rods are attached, and so forth—affects how the kite flies. In this activity, students are given pieces of cardstock of various lengths, hole-punched at regular intervals so they can be attached in different places.



These two "rods" form the frame for a kite at the kite factory. By changing the angle at which the sticks are held and the places where the sticks are attached, students discover different properties of quadrilaterals.

Students are challenged to make conjectures and use precise language to describe their findings about which diagonals result in which quadrilaterals. They can discover properties unique to certain quadrilaterals, such as the fact that diagonals that are perpendicular bisectors of each other imply the quadrilateral is a rhombus. To see videos of this lesson being implemented in a high school classroom, visit http://www.insidemathematics.org/ (accessed March 26, 2015).

Similarity, Right Triangle Trigonometry, and Proof

Understand similarity in terms of similarity transformations

M.2HS.39

Verify experimentally the properties of dilations given by a center and a scale factor.

a. A dilation takes a line not passing through the center of the dilation to a parallel line and leaves a line passing through the center unchanged.
b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

M.2HS.40
Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

M.2HS.41
Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

Prove theorems involving similarity.
M.2HS.45
Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally and conversely; the Pythagorean Theorem proved using triangle similarity.

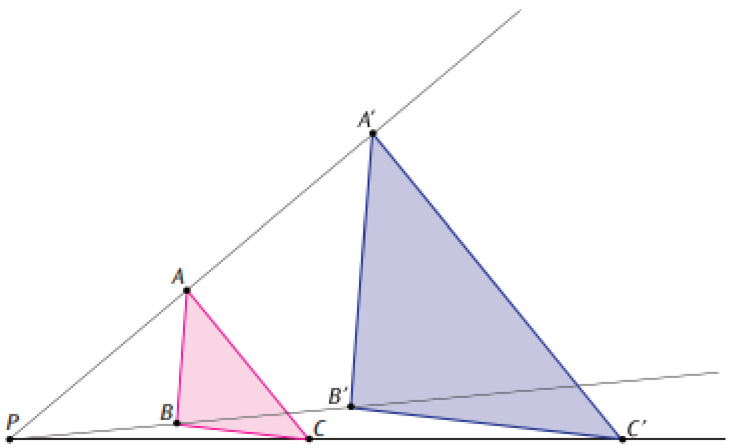
M.2HS.46
Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Because right triangles and triangle relationships play such an important role in applications and future mathematics learning, they are given a prominent role in the geometry conceptual category. A discussion of *similarity* is necessary first—and again, a more precise mathematical definition of similarity is given in the higher mathematics standards. Students work with *dilations* as a transformation in the grade-eight standards; now they explore the properties of dilations in more detail and develop an understanding of the notion of *scale factor* (**M.2HS.39**). Whereas it is common to say that objects that are similar have "the same shape," the new definition for two objects being similar is that there is a sequence of *similarity transformations*—translation, rotation, reflection, or dilation—that maps one object exactly onto the other. Standards **M.2HS.40** and **M.2HS.41** call for students to explore the consequences of two triangles being similar: that they have congruent angles and that their side lengths are in the same proportion. This new understanding gives rise to more results that are encapsulated in standards **M.2HS.45** and **M.2HS.46**.

Example: Experimenting with Dilations	**M.2HS.40– M.2HS.41**
Students are given opportunities to experiment with dilations and determine how they affect planar objects. Students first make sense of the definition of a *dilation of scale factor* $k > 0$ *with center* P as the transformation that moves a point A along the ray $\overline{PA}$ to a	

new point A', so that $|\overline{PA'}| = k \bullet |\overline{PA}|$. For example, using a ruler, students apply the dilation of scale factor 2.5 with center P to the points A, B, and C illustrated below. Once this is done, the students consider the two triangles ΔABC and $\Delta A'B'C'$, and they discover that the lengths of the corresponding sides of the triangles have the same ratio dictated by the scale factor as shown by standard **M.2HS.40**.

Students learn that parallel lines are taken to parallel lines by dilations; thus, corresponding segments of ΔABC and $\Delta A'B'C'$ are parallel. After students have proved results about parallel lines intersected by a transversal, they can deduce that the angles of the triangles are congruent. Through experimentation, they see that the congruence of corresponding angles is a necessary and sufficient condition for the triangles to be similar, leading to the AA criterion for triangle similarity (see standard **M.2HS.41**).



Similarity, Right Triangle Trigonometry, and Proof

Define trigonometric ratios and solve problems involving right triangles.

M.2HS.48
Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

M.2HS.49
Explain and use the relationship between the sine and cosine of complementary angles.

M.2HS.50
Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

Once the angle–angle (AA) similarity criterion for triangles is established, it follows that any two *right* triangles ΔABC and ΔDEF with at least one pair of non-right angles congruent (say $\angle A \cong \angle D$) is similar, since the right angles are obviously congruent (say $\angle B \cong \angle E$). By similarity, the corresponding sides of the triangles are in proportion:

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Notice the first and third expressions in the statement of equality above can be rearranged to yield that

$$\frac{AB}{AC} = \frac{DE}{DF}$$

Since the triangles in question are arbitrary, this implies that for any right triangle with an angle congruent to $\angle A$, the ratio of the side adjacent to $\angle A$ and the hypotenuse of the triangle is a certain constant. This allows us to unambiguously define the *sine* of $\angle A$, denoted by $\sin A$, as this ratio. In this way, students come to understand the trigonometric functions as relationships completely determined by angles (**M.2HS.48**). They further their understanding of these functions by investigating relationships between sine, cosine, and tangent; by exploring the relationship between the sine and cosine of complementary angles; and by applying their knowledge of right triangles to real-world situations (**MHM4**), such as in the example below which illustrates standards **M.2HS.48– M.2HS.50**. Experience in working with many different triangles, finding their measurements, and computing ratios of the measurements will help students understand the basics of the trigonometric functions.

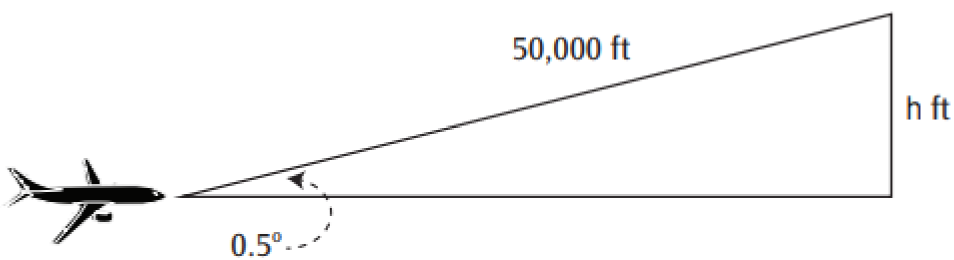
Example: Using Trigonometric Relationships **M.2HS.48– M.2HS.50**

Airplanes that travel at high speeds and low elevations often have onboard radar systems to detect possible obstacles in their path. The radar can determine the range of an obstacle and the angle of elevation to the top of the obstacle. Suppose that the radar detects a tower that is 50,000 feet away, with an angle of elevation of 0.5 degrees. By how many feet must the plane rise in order to pass above the tower?

Solution:

The sketch below shows that there is a right triangle with a hypotenuse of 50,000 (ft) and smallest angle 0.5 (degrees). To find the side opposite this angle, which represents the minimum height the plane should rise, students would use

$$\sin 0.5° = \frac{h}{50{,}000}, \text{ so that } h = (50{,}000 \text{ ft}) \sin 0.5° \approx 436.33 \text{ ft}.$$



Circles With and Without Coordinates

Understand and apply theorems about circles.

M.2HS.52

Prove that all circles are similar.

M.2HS.53

Identify and describe relationships among inscribed angles, radii and chords. Include the relationship between central, inscribed and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

M.2HS.54

Construct the inscribed and circumscribed circles of a triangle and prove properties of angles for a quadrilateral inscribed in a circle.

M.2HS.55(+)

Construct a tangent line from a point outside a given circle to the circle.

Find arc lengths and areas of sectors of circles.

M.2HS.56

Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector. Instructional Note: Emphasize the similarity of all circles. Note that by similarity of sectors with the same central angle, arc lengths are proportional to the radius. Use this as a basis for introducing radian as a unit of measure. It is not intended that it be applied to the development of circular trigonometry in this course.

Students can extend their understanding of the usefulness of similarity transformations by investigating circles (**M.2HS.52**). For instance, students can reason that any two circles are similar by describing precisely how to transform one into the other.

Example	**M.2HS.52**

Students can show that the two circles C and D given by the equations below are similar.

$$C: (x-1)^2 + (y-4)^2 = 9$$
$$D: (x+2)^2 + (y-1)^2 = 25$$

Solution:

The centers of the circles are (1, 4) and (-2, 1), respectively, so the first step is to translate the center of circle C to the center of circle D using the translation $T(x, y) = (x-3, y-3)$. The second and final step is to dilate from the point (-2, 1) by a scale factor of $\frac{5}{3}$, since the radius of circle C is 3 and the radius of circle D is 5.

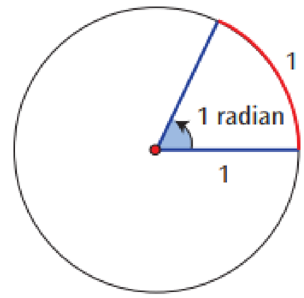
Students continue investigating properties of circles and relationships among angles, radii, and chords as given by standards **M.2HS.53– M.2HS.55**.

Another important application of the concept of similarity is the definition of the radian measure of an angle. Students can derive this result in the following way: given a sector of a circle C of radius r and central angle α, and a sector of a circle D of radius s and central angle also α, it stands to reason that because these sectors are similar,

$$\frac{\text{length of arc on circle } C}{r} = \frac{\text{length of arc on circle } D}{s}$$

Therefore, as with the definition of trigonometric functions, there is a constant m such that for an arc subtended by an angle α on any circle:

$$\frac{\text{length of arc subtennded by angle } \alpha}{\text{radius of the circle}} = m.$$



This constant of proportionality is the *radian measure* of angle α. It follows that an angle that subtends an arc on a circle that is the same length as the radius measures 1 radian. By investigating circles of different sizes, using string to measure arcs subtended by the same angle, and finding the ratios described above, students can apply their proportional-reasoning skills to discover this constant ratio, thereby developing an understanding of the definition of *radian measure*.

Similarity, Right Triangle Trigonometry, and Proof
Use coordinates to prove simple geometric theorems algebraically. **M.2HS.47** Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

Circles With and Without Coordinates
Translate between the geometric description and the equation for a conic section. **M.2HS.57** Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation. Instructional Note: Connect the equations of circles and parabolas to prior work with quadratic equations. **M.2HS.58** Derive the equation of a parabola given the focus and directrix. Instructional Note: The directrix should be parallel to a coordinate axis. **Use coordinates to prove simple geometric theorems algebraically.** **M.2HS.59** Use coordinates to prove simple geometric theorems algebraically. (e.g., Prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point $(1, \sqrt{3})$ lies on the circle centered at the origin and containing the point $(0, 2)$.) Instructional Note: Include simple proofs involving circles.

The largest intersection of algebraic and geometric concepts occurs here, wherein two-dimensional shapes are represented on a coordinate system and can be described using algebraic equations and inequalities. Readers will be familiar with the derivation of the equation of a circle by the Pythagorean Theorem and the definition of a circle (**M.2HS.57**): given that a circle consists of all points (x, y) that are at a distance $r > 0$ from a fixed center (h, k), students see that $\sqrt{(x-h)^2 + (y-k)^2} = r$ for any point lying on the circle, so that $(x-h)^2 + (y-k)^2 = r^2$ determines this circle. Students can derive this equation and flexibly change an equation into this form by completing the square as necessary. By understanding the derivation of this equation, students develop a clear meaning of the variables h, k, and r. Standard **M.2HS.58** calls for students to do the same for the definition of a parabola in terms of a focus and directrix. Numerous resources are available for application problems involving parabolas and should be explored to connect the geometric and algebraic aspects of these curves.

Circles With and Without Coordinates

Explain volume formulas and use them to solve problems.

M.2HS.60

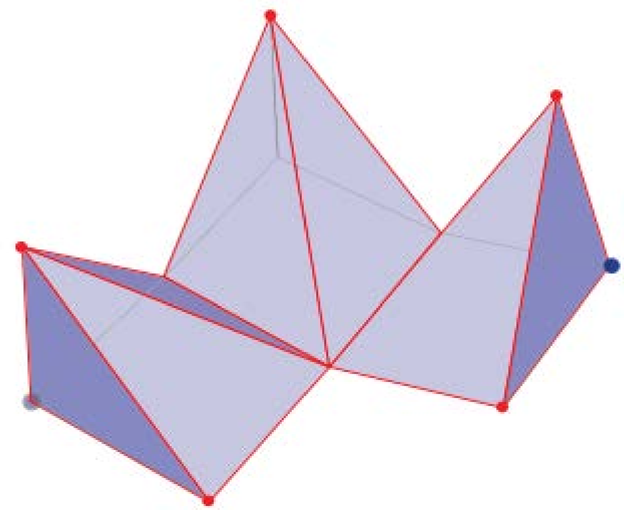
Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle and informal limit arguments. Instructional Note: Informal arguments for area and volume formulas can make use of the way in which area and volume scale under similarity transformations: when one figure in the plane results from another by applying a similarity transformation with scale factor k, its area is k^2 times the area of the first.

M.2HS.61

Use volume formulas for cylinders, pyramids, cones and spheres to solve problems. Volumes of solid figures scale by k^3 under a similarity transformation with scale factor k.

The ability to visualize two- and three-dimensional shapes is a useful skill. This group of standards addresses that skill and includes understanding and using volume and area formulas for curved objects. Students also have the opportunity to make use of the concept of a *limiting process*—an idea that plays a large role in calculus and advanced mathematics courses—when they investigate the formula for the area of a circle. By experimenting with grids of finer and finer mesh, students can repeatedly approximate the area of a unit circle and thereby get a better and better approximation for the irrational number π. They also dissect shapes and make arguments based on these dissections. For instance, as shown in the figure below, a cube can be dissected into three congruent pyramids, which can lend weight to the formula that the volume of a pyramid of base area B and height h is $\frac{1}{3}Bh$ (**MHM2**).

Three Congruent Pyramids that Form a Cube



Source: Park City Mathematics Institute 2013.

Conceptual Category: Statistics and Probability

In grades seven and eight, students learn some basics of probability, including chance processes, probability models, and sample spaces. In higher mathematics, the relative frequency approach to probability is extended to conditional probability and independence, rules of probability and their use in finding probabilities of compound events, and the use of probability distributions to solve problems involving expected value (UA Progressions Documents 2012d, 13). Building on probability concepts that were developed in grades six through eight, students use the language of set theory to expand their ability to compute and interpret theoretical and experimental probabilities for compound events, attending to mutually exclusive events, independent events, and conditional probability. Students should make use of geometric probability models wherever possible. They use probability to make informed decisions (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010a).

Applications of Probability
Understand independence and conditional probability and use them to interpret data.
M.2HS.28
Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes or as unions, intersections or complements of other events ("or," "and," "not").

M.2HS.29
Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities and use this characterization to determine if they are independent.

M.2HS.30
Understand the conditional probability of A given B as P(A and B)/P(B), and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.

M.2HS.31
Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. (e.g., Collect data from a random sample of students in your school on their favorite subject among math, science and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.) Instructional Note: Build on work with two-way tables from Mathematics I to develop understanding of conditional probability and independence.

M.2HS.32
Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. (e.g., Compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.)

Use the rules of probability to compute probabilities of compound events in a uniform probability model.

M.2HS.33
Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A and interpret the answer in terms of the model.

M.2HS.34
Apply the Addition Rule, P(A or B) = P(A) + P(B) – P(A and B), and interpret the answer in terms of the model.

M.2HS.35(+)
Apply the general Multiplication Rule in a uniform probability model, P(A and B) = P(A)P(B|A) = P(B)P(A|B), and interpret the answer in terms of the model.

M.2HS.36(+)
Use permutations and combinations to compute probabilities of compound events and solve problems.

Standards identified with (+) are included to increase coherence but are not necessarily expected to be addressed on high stakes assessments.

To develop student understanding of conditional probability, students should experience two types of problems: those in which the uniform probabilities attached to outcomes lead to independence of the outcomes, and those in which they do not (**M.2HS.28– M.2HS.30**). The following examples illustrate these two distinct possibilities.

Example: Guessing on a True—False Quiz **M.2HS.28– M.2HS.30**

If there are four true-or-false questions on a quiz, then the possible outcomes based on guessing on each question may be arranged as in the table below:

Possible outcomes: Guessing on four true-false questions

Number correct	Out-comes	Number correct	Out-comes	Number correct	Out-comes
4	CCCC	2	CCII	1	CIII
3	ICCC	2	CICI	1	ICII
3	CICC	2	CIIC	1	IICI
3	CCIC	2	ICCI	1	IIIC
3	CCCI	2	ICIC	0	IIII
		2	IICC		

C indicates a correct answer; I indicates an incorrect answer.

By counting outcomes, one can find various probabilities. For example:

$$P(C \text{ on first question}) = \frac{1}{2}$$

and

$$P(C \text{ on second question}) = \frac{1}{2}$$

Noticing that $P[(C \text{ on first}) \text{ AND } (C \text{ on second})] = \frac{4}{16} = \frac{1}{2} \bullet \frac{1}{2}$ shows that the two events—getting the first question correct and the second question correct—are independent.

Adapted from UA Progressions Documents 2012d.

Example: Work-Group Leaders	**M.2HS.28– M.2HS.30**

Suppose a five-person work group consisting of three girls (April, Briana, and Cyndi) and two boys (Daniel and Ernesto) wants to randomly choose two people to lead the group. The first person is the discussion leader and the second is the recorder, so order is important in selecting the leadership team. In the table below, "A" represents April, "B" represents Briana, "C" represents Cyndi, "D" represents Daniel, and "E" represents Ernesto. There are 20 outcomes for this situation:

Selecting two students from three girls and two boys		
Number of girls	Outcomes	
2	AB	BA
2	AC	CA
2	BC	CB
1	AD	DA
1	AE	EA
1	BD	DB
1	BE	EB
1	CD	DC
1	CE	EC
0	DE	ED

Notice that the probability of selecting two girls as the leaders is as follows:

$$P(\text{two girls chosen}) = \frac{6}{20} = \frac{3}{10}$$

whereas

$$P(\text{girl selected on first draw}) = \frac{12}{20} = \frac{3}{5}$$

and

$$P(\text{girl selected on second draw}) = \frac{3}{5}$$

But since $\frac{3}{5} \bullet \frac{3}{5} \neq \frac{3}{10}$, the two events are not independent.

One can also use the conditional-probability perspective to show that these events are not independent.

Since $P(\text{girl on second} \mid \text{girl on first}) = \frac{6}{12} = \frac{1}{2}$

and

$P(\text{girl selected on second}) = \frac{3}{5}$,

These events are seen to be dependent.

Adapted from UA Progressions Documents 2012d.

Students also explore finding probabilities of compound events, as in standards **M.2HS.33–M.2HS.36,** by using the Addition Rule ($P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$) and the general Multiplication Rule ($P(A \text{ and } B) = P(A) \bullet P(B|A) = P(B) \bullet P(A|B)$). A simple experiment in which students roll two number cubes and tabulate the possible outcomes can shed light on these formulas before they are extended to application problems.

Example **M.2HS.33– M.2HS.36**

On April 15, 1912, the RMS *Titanic* rapidly sank in the Atlantic Ocean after hitting an iceberg. Only 710 of the ship's 2,204 passengers and crew members survived. Some believe that the rescue procedures favored the wealthier first-class passengers. Data on survival of passengers are summarized in the table at the end of this example, and these data will be used to investigate the validity of such claims. Students can use the fact that two events A and B are independent if $P(A|B) = P(A) \bullet P(B)$. A represents the event that a passenger survived, and B represents the event that the passenger was in first class. The conditional probability $P(A|B)$ is compared with the probability $P(A)$.

For a first-class passenger, the probability of surviving is the fraction of all first-class passengers who survived. That is, the sample space is restricted to include only first-class passengers to obtain:

$$P(A|B) = \frac{202}{325} \approx 0.622$$

The probability that a passenger survived is the number of all passengers who survived divided by the total number of passengers:

$$P(A) = \frac{498}{1316} \approx 0.378$$

Since $0.622 \neq 0.378$, the two given events are not independent. Moreover, it can be said that being a passenger in first class did increase the chances of surviving the accident.

Students can be challenged to further investigate where similar reasoning would apply today. For example, what are similar statistics for Hurricane Katrina, and what would a similar analysis conclude about the distribution of damages? (**MMH4**)

***Titanic* passengers**	**Survived**	**Did not survive**	**Total**

First-class	202	123	325
Second-class	118	167	285
Third-class	178	528	706
Total passengers	498	818	1,316

Adapted from Illustrative Mathematics 2013q.

Applications of Probability

Use probability to evaluate outcomes of decisions.

M.2HS.37(+)

Use probabilities to make fair decisions (e.g., drawing by lots or using a random number generator).

M.2HS.38(+)

Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, and/or pulling a hockey goalie at the end of a game). Instructional Note: This unit sets the stage for work in Mathematics III, where the ideas of statistical inference are introduced. Evaluating the risks associated with conclusions drawn from sample data (i.e., incomplete information) requires an understanding of probability concepts.

Standards **M.2HS.37** and **M.2HS.38** involve students' use of probability models and probability experiments to make decisions. These standards set the stage for more advanced work in Mathematics III, where the ideas of statistical inference are introduced. See the University of Arizona Progressions document titled "High School Statistics and Probability" for further explanation and examples: http://ime.math.arizona.edu/progressions/ (UA Progressions Documents 2012d [accessed April 6, 2015]).

Common Misconceptions – By Domain

Number and Quantity

- Students often have difficulty differentiating between finding the solution to an equation such as $x^2 = 25$ and taking the square root of a number. The equation $x^2 = 25$ has two solutions, 5 and -5. Expressions such as $\sqrt{3^2}$ or $\sqrt{(-7)^2}$ ask for the "principal" square root. As a result, $\sqrt{3^2}$ = 3 and $\sqrt{(-7)^2}$ = 7

- Student confusion in differentiating between expressions involving the square of a negative number and the negative of a square, often extend to square roots. Students may need help in appreciating the Order of Operations in differentiating between the expressions.

$(-5)^2$ → (-5)(-5) → 25
$-(5)^2$ → - (5)(5) → -25
$\sqrt{(-5)^2}$ → $\sqrt{25}$ → 5
$-\sqrt{5^2}$ → - $\sqrt{25}$ → -5
$\sqrt{(5)^2}$ → $\sqrt{-25}$ → 5

- Students may have difficulty when negative exponents involve fractions. Students may incorrectly believe that $16^{-\frac{1}{2}}$ means -4 rather than $\frac{1}{4}$.

$$16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$$

Algebra

- Students may have difficulty distinguishing between the terms of an expression and the variable of an expression. For example, students may reason that the trinomial $3a + b + 5cd$ has four terms because the student is counting variable rather than terms. Often students do not recognize a constant as a term and incorrectly identify the trinomial $4x - 2y + 7$ as having two terms.

- Students who rely solely on procedures may believe they need multiply the binomials in an equation such as $(x + 4)(x - 3) = 0$ and then factor the resulting expression to solve for the zeros. Students need to develop an understanding of the concept of the Zero Factor Property. Students may experience similar confusion in applying the Zero Factor Property in solving equations such as $5(x - 7)(x + 1) = 0$ or $6x(x - 7) = 0$.

- Students who mistakenly believe that $\sqrt{x^2 + y^2}$ is equivalent to $x + y$, may deduce this from the misconception that $(x + y)^2$ is equivalent to $x^2 + y^2$.

- Students who mistakenly equate $(x + y)^2$ with $x^2 + y^2$ or equate $(x - y)^2$ with $x^2 - y^2$ have misconceptions regarding the concept of finding the square of a number or expression. These misconceptions may stem from students' prior difficulty in recognizing that $(-5)^2$ → (-5)(-5) →25 or that $-(5)^2$ → –(5)(5) → -25.

- Students who incorrectly equate rational expressions such as $\frac{x^2 - 6x + 9}{x - 3}$ with $x - 2x + 9$ or $x - 6x - 3$ or $x^2 + 2x + 9$, etc., have difficulty distinguishing between terms and factors.

- Students may demonstrate difficulties in solving the equation $\sqrt{x-1}$ = $x-7$. Students may appropriately decide to square both sides of the equation and write $(\sqrt{x-1}\,)^2$ = $(x-7)^2$ Student error may arise in writing $(\sqrt{x-1}\,)^2$ = x^2 - 7^2, demonstrating misconceptions in squaring binomials (or multiplying) binomials.

- The procedure for solving equations and inequalities are so similar. As a result, students often forget to attend to precision when multiplying and dividing by a negative number when solving inequalities.   Students may correctly solve equations such as $-2x = 6$ and find its solution to be $x = -3$.   Misconceptions arise when students equate solving the equation with a similar inequality $-2x \le 6$ and incorrectly determine that $x \le -3$. The misconception can be addressed by encouraging students to verify solutions to equations and to inequalities.

- Students often solve rational and radical equations without checking to determine if any solution may be erroneous. Students should be encouraged to verify solutions.

Functions

- Students may erroneously interpret the notation $g(3)$ to mean "g $times$ 3".

- Students often believe that all functions must use the symbols $f, x,$ and y.

- When graphing students may confuse the parts of the slope-equation form of a linear equation. Students may incorrectly determine that the function $g(x) = 2x + \frac{3}{5}$ has a slope of $\frac{3}{5}$ and a y-intercept of 2. Students may be relying on a procedural understanding that the y-intercept is always an integer and the slope or $\frac{rise}{run}$ must be a fraction.

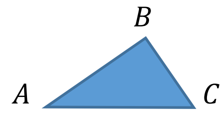
- Students often incorrectly assume that the function $f(x + k)$ where $k > 0$ will result in a horizontal shift of the graph k units to the right.

Geometry

- Students often think of congruence as "figures with the same shape and size." While this understanding is not incorrect, it is important to that the continually emphasize the link of congruence with rigid motions and show that rigid motions do in fact produce "figures with the same shape and size."

- Students may look at scale factor as the distance that is added on to the original distance. Dilations where the center of dilation is a vertex of a figure can prove challenging because the sides of the pre-image and image overlap.

- Students may have difficulty identifying the relevant sides in relationship to a given angle, especially If the triangle is not depicted in a typical position where its legs are horizontal and vertical. For example, given $\Delta\, ABC$ below with right angle $B,$ students may struggle to understand that $\sin A = \frac{BC}{AC}$ and that $\sin A \neq \frac{BC}{AB}$.



- Students may have difficulty differentiating between the meaning of the statements $\Delta\, ABC \cong \Delta\, DEF$ and $\Delta\, ABC \sim \Delta\, DEF$. While $\Delta\, ABC \cong \Delta\, DEF$ implies that AB = DE, $\Delta\, ABC \sim \Delta\, DEF$ does not imply that AB = DE.

- Students often have difficulty differentiating among $\overleftarrow{EF}$, $\overrightarrow{EF}$, $\overleftrightarrow{EF}$, and $\overline{EF}$.

Statistics and Probability

- Students often use the word *outlier* inaccurately, failing to verify that it satisfies the necessary conditions. Using terms such as *unusual feature* or *data point* can help students avoid using the term outlier when it is not appropriate.

- Students commonly believe that any data that is collected should follow a normal distribution.

- Students need to understand that association does not necessarily provide evidence for cause and effect.

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