West Virginia math4life

# Educators' Guide for Mathematics

## *Mathematics III*

West Virginia DEPARTMENT OF EDUCATION

# Mathematics III

In the Mathematics III course, students expand their repertoire of functions to include polynomial, rational, and radical functions. They also expand their study of right-triangle trigonometry to include general triangles. And, finally, students bring together all of their experience with functions and geometry to create models and solve contextual problems. The courses in the Integrated Pathway follow the structure introduced in the K–8 grade levels of the West Virginia College- and Career- Readiness Standards; they present mathematics as a coherent subject and blend standards from different conceptual categories.

The standards in the integrated Mathematics III course come from the following conceptual categories: modeling, functions, number and quantity, algebra, geometry, and statistics and probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

Math III LA course does not include the (+) standards.
Math III STEM course includes standards identified by (+) sign.
Math III TR course (Technical Readiness) includes standards identified by (*).
Math IV TR course (Technical Readiness) includes standards identified by (^).

Math III Technical Readiness and Math IV Technical Readiness are course options (for juniors and seniors) built for the mathematics content of Math III through integration of career clusters. These courses integrate academics with hands-on career content. The collaborative teaching model is recommended and should be based at our Career and Technical Education (CTE) centers. The involvement of a highly qualified Mathematics teacher and certified CTE teachers will ensure a rich, authentic, and respectful environment for delivery of the academics in “real world” scenarios.

## What Students Learn in Mathematics III

In Mathematics III, students understand the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. They connect multiplication of polynomials with multiplication of multi-digit integers and division of polynomials with long division of integers. Students identify zeros of polynomials and make connections between zeros of polynomials and solutions of polynomial equations. Their work on polynomial expressions culminates with the Fundamental Theorem of

Algebra. Rational numbers extend the arithmetic of integers by allowing division by all numbers except 0. Similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme of working with rational expressions is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect, regardless of the type of the underlying functions.

Students develop the Laws of Sines and Cosines in order to find missing measures of general (not necessarily right) triangles. They are able to distinguish whether three given measures (angles or sides) define 0, 1, 2, or infinitely many triangles. This discussion of general triangles opens up the idea of trigonometry applied beyond the right triangle—that is, at least to obtuse angles. Students build on this idea to develop the notion of radian measure for angles and extend the domain of the trigonometric functions to all real numbers. They apply this knowledge to model simple periodic phenomena.

Students see how the visual displays and summary statistics they learned in previous grade levels or courses relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and recognize the role that randomness and careful design play in the conclusions that may be drawn.

Finally, students in Mathematics III extend their understanding of modeling: they identify appropriate types of functions to model a situation, adjust parameters to improve the model, and compare models by analyzing appropriateness of fit and by making judgments about the domain over which a model is a good fit. The description of modeling as “the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions” (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010e) is one of the main themes of this course. The discussion about modeling and the diagram of the modeling cycle that appear in this chapter should be considered when students apply knowledge of functions, statistics, and geometry in a modeling context.

## Examples of Key Advances from Mathematics II

- Students begin to see polynomials as a system analogous to the integers that they can add, subtract, multiply, and so forth. Subsequently, polynomials can be extended to rational expressions, which are analogous to rational numbers.
- Students extend their knowledge of linear, exponential, and quadratic functions to include a much broader range of classes of functions.
- Students begin to examine the role of randomization in statistical design.

## Connecting Mathematical Habits of Mind and Content

The Mathematical Habits of Mind (**MHM**) apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, relevant, and meaningful subject. The Mathematical Habits of Mind represent a picture of what it looks like for students to do mathematics and, to the extent possible, content instruction should include attention to appropriate practice standards. The Mathematics III course offers ample opportunities for students to engage with each Mathematical Habit of Mind; the following table offers some examples.

**Mathematical Habits of Mind—Explanation & Examples for Mathematics III**

| Mathematical Habits of Mind | Explanation and Examples |
|---|---|
| **MHM1**<br>Make sense of problems and persevere in solving them. | Students apply their understanding of various functions to real-world problems. They approach complex mathematics problems and break them down into smaller problems, synthesizing the results when presenting solutions. |
| **MHM2**<br>Reason abstractly and quantitatively. | Students deepen their understanding of variables—for example, by understanding that changing the values of the parameters in the expression $A\sin(Bx + C) + D$ has consequences for the graph of the function. They interpret these parameters in a real-world context. |
| **MHM3**<br>Construct viable arguments and critique the reasoning of others. | Students continue to reason through the solution of an equation and justify their reasoning to their peers. Students defend their choice of a function when modeling a real-world situation. |
| **MHM4**<br>Model with mathematics. | Students apply their new mathematical understanding to real-world problems, making use of their expanding repertoire of functions in modeling. Students also discover |

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| | mathematics through experimentation and examination of patterns in data from real-world contexts. |
| **MHM5**<br>Use appropriate tools strategically. | Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions. |
| **MHM6**<br>Attend to precision. | Students make note of the precise definition of *complex number*, understanding that real numbers are a subset of complex numbers. They pay attention to units in real-world problems and use unit analysis as a method for verifying their answers. |
| **MHM7**<br>Look for and make use of structure. | Students understand polynomials and rational numbers as sets of mathematical objects that have particular operations and properties. They understand the periodicity of sine and cosine and use these functions to model periodic phenomena. |
| **MHM8**<br>Look for and express regularity in repeated reasoning. | Students observe patterns in geometric sums—for example, that the first several sums of the form $\sum_{k=0}^{n} 2^k$ can be written as follows:<br>$1 = 2^1 - 1$<br>$1 + 2 = 2^2 - 1$<br>$1 + 2 + 4 = 2^3 - 1$<br>$1 + 2 + 4 + 8 = 2^4 - 1$<br>Students use this observation to make a conjecture about any such sum. |

**MHM4** holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. Examples where specific Mathematical Habits of Mind can be implemented in the Mathematics III standards are noted in parentheses, with the standard(s) also listed.

## Mathematics III Content Standards, by Conceptual Category

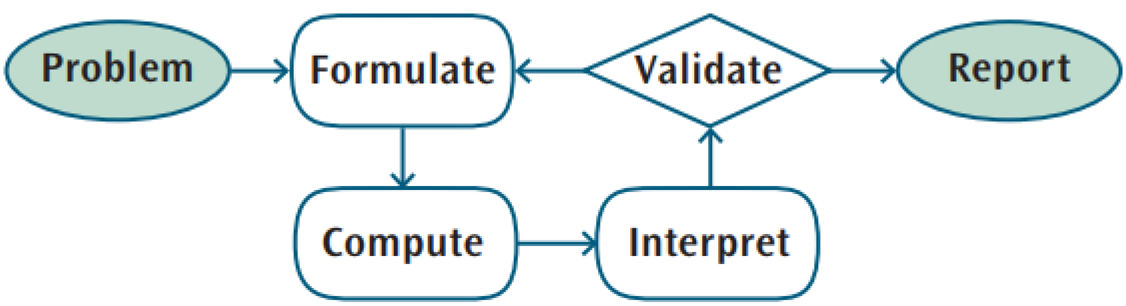
The Mathematics III course is organized by domains, clusters, and then standards. The overall purpose and progression of the standards included in Mathematics III are described below, according to each conceptual category. Standards that are considered new for secondary-grades teachers are discussed more thoroughly than other standards.

## Conceptual Category: Modeling

Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them and then constructing mathematics in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise (e.g., which of the quantities present in this situation are known, and which are unknown?). Students need to decide on a solution path that may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They use previously derived models (e.g., linear functions) as well as new formulas or functions that apply. In addition, students may discover that answering their question requires solving an equation and knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, or the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see the following figure. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

**Modeling Cycle**




The examples in this chapter are framed as much as possible to illustrate the concept of mathematical modeling. The important ideas surrounding rational functions, graphing, solving equations, and rates of change are explored through this lens.

## Conceptual Category: Functions

The standards in the Functions conceptual category can serve as motivation for the study of standards in the other Mathematics III conceptual categories. Students have already worked with equations in which they have to "solve for $x$" as a search for the input of a function $f$ that gives a specified output; solving the equation amounts to undoing the work of the function. The types of functions that students encounter in Mathematics III have new properties. Students previously learned that quadratic functions exhibit different behavior from linear and exponential functions; now they investigate polynomial, rational, and trigonometric functions in greater generality. As in the Mathematics II course, students must discover new techniques for solving the equations they encounter. Students see how rational functions can model real-world phenomena, in particular in instances of inverse variation ($x \bullet y = k$, $k$ a constant), and how trigonometric functions can model periodic phenomena. In general, functions describe how two quantities are related in a precise way and can be used to make predictions and generalizations, keeping true to the emphasis on modeling that occurs in higher mathematics. As stated in the University of Arizona (UA) Progressions Documents, "students should develop ways of thinking that are general and allow them to approach any function, work with it, and understand how it behaves, rather than see each function as a completely different animal in the bestiary" (UA Progressions Documents 2013c, 7).

### Polynomial, Rational, and Radical Relationships

**Analyze functions using different representations.**

**M.3HS.24 (*,^)**

Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. Graph polynomial functions, identifying zeros when suitable factorizations are available and showing end behavior. Instructional Note: Relate to the relationship between zeros of quadratic functions and their factored forms.

### Mathematical Modeling

**Interpret functions that arise in applications in terms of a context.**

**M.3HS.35 (*)**

For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. Instructional Note: Emphasize the selection of a model function based on behavior of data and context.

**M.3HS.36 (*)**
Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. (e.g., If the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.) Instructional Note: Emphasize the selection of a model function based on behavior of data and context.

**M.3HS.37 (*)**
Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.
Instructional Note: Emphasize the selection of a model function based on behavior of data and context.

**Analyze functions using different representations.**
**M.3HS.38 (*,^)**
Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

a. Graph square root, cube root and piecewise-defined functions, including step functions and absolute value functions.
b. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline and amplitude.

Instructional Note: Focus on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate.

**M.3HS.39 (*,^)**
Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. Instructional Note: Focus on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate.

**M.3HS.40 (*,^)**
Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). (e.g., Given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.) Instructional Note: Focus on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate.

As in Mathematics II, students work with functions that model data and choose an appropriate model function by considering the context that produced the data. Students' ability to recognize rates of change, growth and decay, end behavior, roots, and other characteristics of functions becomes more sophisticated; they use this expanding repertoire of families of functions to inform their choices of models. This group of standards focuses on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate (**M.3HS.35–40**). The following example illustrates some of these standards. (Note that only sine, cosine, and tangent are treated in Mathematics III.)

**Example: The Juice Can** **M.3HS.35– M.3HS.40**

Students are asked to find the minimal surface area of a cylindrical can of a fixed volume. The surface area is represented in units of square centimeters ($cm^2$), the radius in units of centimeters (cm), and the volume is fixed at 355 milliliters (ml), or 355 $cm^3$. Students can find the surface area of this can as a function of the radius:

$$S(r) = \frac{2(355)}{r} + 2\pi r^2$$

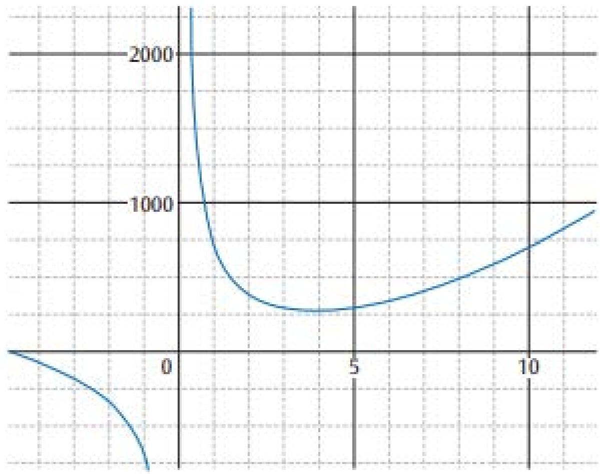
(See The Juice-Can Equation example that appears in the Algebra conceptual category of this chapter.)

This representation allows students to examine several things. First, a table of values will provide a hint about what the minimal surface area is. The table below lists several values for $S$ based on $r$:

| $r$ (cm) | $S$ ($cm^2$) |
|---|---|
| 0.5 | 1421.6 |
| 1.0 | 716.3 |
| 1.5 | 487.5 |
| 2.0 | 380.1 |
| 2.5 | 323. |
| 3.0 | 293.2 |
| 3.5 | 279.8 |
| 4.0 | 278.0 |
| 4.5 | 284.9 |
| 5.0 | 299.0 |
| 5.5 | 319.1 |
| 6.0 | 344.4 |
| 6.5 | 374.6 |
| 7.0 | 409.1 |

| | |
|---|---|
| 7.5 | 447.9 |
| 8.0 | 490.7 |

The data suggest that the minimal surface area occurs when the radius of the base of the juice can is between 3.5 and 4.5 centimeters. Successive approximations using r values of between these values will yield a better estimate. But how can students be sure that the minimum is truly located here? A graph of $S(r)$ provides a clue:




Furthermore, students can deduce that as $r$ gets smaller, the term $\frac{2(355)}{r}$ gets larger and larger, while the term $2\pi r$ gets smaller and smaller, and that the reverse is true as $r$ grows larger, so that there is truly a minimum somewhere in the interval [3.5, 4.5].

Graphs help students reason about rates of change of functions (**M.3HS.37**). In grade eight, students learn that the *rate of change* of a linear function is equal to the slope of the graph of that function. And because the slope of a line is constant, the phrase "rate of change" is clear for linear functions. For non-linear functions, however, rates of change are not constant, and thus average rates of change over an interval are used. For example, for the function $g$ defined for all real numbers by $g(x) = x^2$, the average rate of change from $x = 2$ to $x = 5$ is

$$\frac{g(5) - g(2)}{5 - 2} = \frac{25 - 4}{5 - 2} = \frac{21}{3} = 7.$$

This is the slope of the line containing the points (2,4) and (5,25) on the graph of $g$. If $g(x)$ is interpreted as returning the area of a square of side length $x$, then this calculation means that over this interval the area changes, on average, by 7 square units for each unit increase in the side length of the square (UA Progressions Documents 2013c, 9). Students could investigate similar rates of change over intervals for the Juice-Can problem shown previously.

**Mathematical Modeling**

**Build a function that models a relationship between two quantities.**

**M.3HS.41 (*)**

Write a function that describes a relationship between two quantities. Combine standard function types using arithmetic operations. (e.g., Build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.) Instructional Note: Develop models for more complex or sophisticated situations than in previous courses.

**Build new functions from existing functions.**

**M.3HS.42 (*)**

Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. Instructional Note: Use transformations of functions to find more optimum models as students consider increasingly more complex situations. Note the effect of multiple transformations on a single function and the common effect of each transformation across function types. Include functions defined only by graph.

**M.3HS.43 (*)**

Find inverse functions. Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. (e.g., $f(x) = 2x^3$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$.) Instructional Note: Extend this standard to simple rational, simple radical, and simple exponential functions.

Students in Mathematics III develop models for more complex or sophisticated situations than in previous courses, due to the expansion of the types of functions available to them (**M.3HS.41**). Modeling contexts provide a natural place for students to start building functions with simpler functions as components. Situations in which cooling or heating are considered involve functions that approach a limiting value according to a decaying exponential function. Thus, if the ambient room temperature is 70 degrees Fahrenheit and a cup of tea is made with boiling water at a temperature of 212 degrees Fahrenheit, a student can express the function describing the temperature as a function of time by using the constant function $f(t) = 70$ to represent the ambient room temperature and the exponentially decaying $g(t) = 142e^{-kt}$ function to represent the decaying difference between the temperature of the tea and the temperature of the room, which leads to a function of this form:

$$T(t) = 70 + 142e^{-kt}$$

Students might determine the constant $k$ experimentally (**MHM4, MHM5**).

With standard **M.3HS.43**, students learn that some functions have the property that an input can be recovered from a given output; for example, the equation $f(x) = c$ can be solved for $x$, given that $c$ lies in the range of $f$. Students understand that this is an attempt to "undo" the function, or to "go backwards." Tables and graphs should be used to support student understanding here. This standard dovetails nicely with standard **M.3HS.44** described below and should be taught in progression with it. Students will work more formally with inverse functions in advanced mathematics courses, and so standard **M.3HS.44** should be treated carefully to prepare students for deeper understanding of functions and their inverses.

| Mathematical Modeling |
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| **Construct and compare linear, quadratic, and exponential models and solve problems.** |
| **M.3HS.44 (*)**<br>For exponential models, express as a logarithm the solution to a bct = d where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.<br>Instructional Note:  Consider extending this unit to include the relationship between properties of logarithms and properties of exponents, such as the connection between the properties of exponents and the basic logarithm property that log xy = log x + log y. |

Students work with exponential models in Mathematics II. Based on the graph of the exponential function $f(x) = b^x$, students in Mathematics III can deduce that this function has an inverse—which is called *the logarithm to the base* $b$ and denoted by $g(x) = \log_b x$. The logarithm has the property that $\log_b x = y$ if and only if $b^y = x$. Students find logarithms with bases equal to 2, 10, or by hand and with the assistance of technology (**MHM5**). Students may be encouraged to explore the properties of logarithms (such as $\log_b xy = \log_b x + \log_b y$) and to connect these properties to those of exponents. For example, the property just mentioned comes from the fact that the logarithm is essentially an exponent and that $b^{n+m} = b^n \bullet b^m$. Students solve problems involving exponential functions and logarithms and express their answers by using logarithm notation (**M.3HS.44**). In general, students understand logarithms as functions that *undo* their corresponding exponential functions; instruction should emphasize this relationship.

| Trigonometry of General Triangles and Trigonometric Functions |
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| **Extend the domain of trigonometric functions using the unit circle.**<br>**M.3HS.28 (*)**<br>Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.<br><br>**M.3HS.29 (*)**<br>Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.<br><br>**Model periodic phenomena with trigonometric functions.**<br>**M.3HS.30 (*)**<br>Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. |

This set of standards calls for students to expand their understanding of trigonometric functions, which was first developed in Mathematics II. At first, the trigonometric functions apply only to angles in right triangles; for example, $\sin\theta$, $\cos\theta$, and $\tan\theta$ make sense only for $0 < \theta < \frac{\pi}{2}$. By representing right triangles with hypotenuse 1 in the first quadrant of the plane, students see that $(\cos\theta, \sin\theta)$ represents a point on the unit circle. This leads to a natural way to extend these functions to any value of $\theta$ that remains consistent with the values for acute angles: interpreting $\theta$ as the radian measure of an angle traversed from the point (1, 0) counterclockwise around the unit circle, $\cos\theta$ is taken to be the $x$-coordinate of the point corresponding to this rotation and $\sin\theta$ is the $y$-coordinate of this point. This interpretation of sine and cosine immediately yields the Pythagorean Identity: that $\cos^2\theta + \sin^2\theta = 1$. This basic identity yields other identities through algebraic manipulation and allows students to find values of other trigonometric functions for a given if one value is known (**M.3HS.28– M.3HS.29**).

Students should explore the graphs of trigonometric functions, with attention to the connection between the unit-circle representation of the trigonometric functions and their properties—for example, to illustrate the periodicity of the functions, the relationship between the maximums and minimums of the sine and cosine graphs, zeros, and so forth. Standard **M.3HS.30** calls for students to use trigonometric functions to model periodic phenomena. This is connected to standard **M.3HS.42** (families of functions) where students begin to understand the relationship between the parameters appearing in the general cosine function $f(x) = A \bullet \cos(Bx - C) + D$ (and sine function), as well as the graph and behavior of the function (e.g., amplitude, frequency, line of symmetry). Additionally, students use their understanding of inverse functions to explore the inverse sine, cosine, and tangent functions at a basic level. It is important for students to understand that a function is well defined only when its domain is specified. For example, the general sine function, $\sin x$, defined for all real numbers, does not have an inverse, whereas the function $s(x) = \sin x$, defined only for values of $x$ such that $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, *does* have an inverse function.

| **Example: Modeling Daylight Hours** | **M.3HS.30** |
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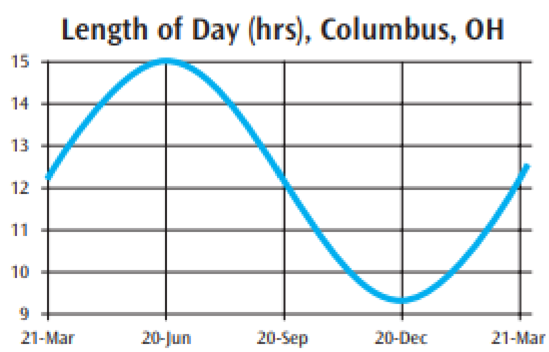
By looking at data for the length of days in Columbus, Ohio, students see that the number of daylight hours is approximately sinusoidal, varying from about 9 hours, 20 minutes on December 21 to about 15 hours on June 21. The average of the maximum and minimum gives the value for the midline, and the amplitude is half the difference of the maximum and minimum. Approximations of these values are set as $A = 12.17$ and $B = 2.83$. With some support, students determine that for the period to be 365 days (per cycle), or for the frequency to be $\frac{1}{365}$ cycles per day, $C = \frac{2\pi}{365}$ and if day 0 corresponds to March 21, no phase shift would be needed, so $D = 0$.

Thus, $f(t) = 12.17 + 2.83 \sin\left(\frac{2\pi t}{365}\right)$ is a function that gives the approximate length of the day for the day of the year March 21. Considering questions such as when to plant a garden (i.e., when there are at least 7 hours of midday sunlight), students might estimate that a 14-hour day is optimal. Students solve $f(t) = 14$ and find that May 1 and August 10 mark this interval of time.




Students can investigate many other trigonometric modeling situations, such as simple predator-prey models, sound waves, and noise-cancellation models.

*Source*: UA Progressions Documents 2013c, 19.

## Conceptual Category: Number and Quantity

Students continue to expand their understanding of the number system by finding complex-number roots when solving quadratic equations. Complex numbers have a practical application, and many phenomena involving real numbers become simpler when real numbers are viewed as a subsystem of complex numbers. As an example, complex solutions of differential equations can present a clear picture of the behavior of real solutions. Students are introduced to this when they study complex solutions of quadratic equations—and when complex numbers are involved, each quadratic polynomial can be expressed as a product of linear factors.

| Polynomial, Rational, and Radical Relationships |
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| **Use complex numbers in polynomial identities and equations.**<br>**M.3HS.10 (+)**<br>Extend polynomial identities to the complex numbers. For example, rewrite x2 + 4 as (x + 2i)(x – 2i). Instructional Note: Build on work with quadratics equations in Mathematics II. Limit to polynomials with real coefficients.<br><br>**M.3HS.11 (+)**<br>Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials. |

Standards **M.3HS.10– M.3HS.11** call for students to continue working with complex numbers as solutions to polynomial equations. This builds on student work with quadratics that started in Mathematics II. For example, students can draw upon the Mathematics III algebra standards and find roots of equations such as $x^3 + 5x^2 + 8x + 6 = 0$. They experiment by using the remainder theorem and find that $x + 3$ is a root, since the polynomial expression evaluated at $x = -3$ is 0. Using polynomial long division or other factoring techniques, students find that

$$x^3 + 5x^2 + 8x + 6 = (x + 3)(x^2 + 2x + 2).$$

They use the quadratics formula to find the roots of the quadratic. $\{-1 + i, -1 - i\}$, and they write

$$x^3 + 5x^2 + 8x + 6 = (x + 3)(x^2 + 2x + 2)$$
$$x^3 + 5x^2 + 8x + 6 = (x + 3)\big(x - (-1 + i)\big)\big(x - (-1 - i)\big).$$

Experimentation with examples of such polynomials and an understanding that the quadratic formula always yields solutions to a quadratic equation help students understand the Fundamental Theorem of Algebra (**M.3HS.11**).

## Conceptual Category: Algebra

Along with the Number and Quantity standards in Mathematics III, the standards from the Algebra domain of the Mathematics III course develop the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multi-digit integers and connect division of polynomials with long division of integers. Similar to the way that rational numbers extend the arithmetic of integers by allowing division by all numbers except 0, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme arises: the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

| Polynomial, Rational, and Radical Relationships |
|---|
| **Interpret the structure of expressions.**<br>**M.3HS.12 (*)**<br>Interpret expressions that represent a quantity in terms of its context.<br>a. Interpret parts of an expression, such as terms, factors, and coefficients.<br>b. Interpret complicated expressions by viewing one or more of their parts as a single entity. (e.g., Interpret P(1 + r)n as the product of P and a factor not depending on P.)<br>Instructional Note: Extend to polynomial and rational expressions.<br><br>**M.3HS.13 (*)**<br>Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. Instructional Note: Extend to polynomial and rational expressions.<br><br>**Write expressions in equivalent forms to solve problems.**<br>**M.3HS.14 (^)**<br>Derive the formula for the sum of a geometric series (when the common ratio is not 1), and use the formula to solve problems. (e.g., Calculate mortgage payments.) Instructional Note: Consider extending to infinite geometric series in curricular implementations of this course description. |

In Mathematics III, students continue to pay attention to the meaning of expressions in context and interpret the parts of an expression by “chunking”—that is, by viewing parts of an expression as a single entity (**M.3HS.12– M.3HS.13**). For example, their facility with using special cases of polynomial factoring allows them to factor more complicated polynomials:

$$x^4 - y^4 = (x^2)^2 - (y^2)^2 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y)$$

Additionally, with their new understanding of complex numbers, students can factor this further into $x^4 - y^4 = (x + iy)(x - iy)(x + y)(x - y)$. In a physics course, students may encounter an expression such as $L_0\sqrt{1 - \frac{v^2}{c^2}}$, which arises in the theory of special relativity. They can see this expression as the product of a constant $L_0$ and a term that is equal to 1 when $v = 0$ and equal to 0 when $v = c$. Furthermore, students might be expected to see this mentally, without having to go through a laborious process of evaluation. This involves combining large-scale structure of the expression—a product of $L_0$ and another term—with the meaning of internal components such as $\frac{v^2}{c^2}$ (UA Progressions Documents 2013b, 4).

By examining the sum of a finite geometric series, students can look for a pattern to justify why the equation for the sum holds: $\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{(1-r)}$. They may derive the formula, either with Proof by Mathematical Induction or by other means (**M.3HS.14**).

| **Example: Sum of a Geometric Series** | **M.3HS.14** |
|---|---|

Students should investigate several concrete examples of finite geometric series (with $r \neq 1$) and use spreadsheet software to investigate growth in the sums and patterns that arise (**MHM5, MHM8**). Geometric series have applications in several areas, including calculating mortgage payments, calculating totals for annual investments such as retirement accounts, finding total payout amounts for lottery winners, and more (**MHM4**). In general, a finite geometric series has this form:

$$\sum_{k=0}^{n} ar^k = a(1 + r + r^2 + \ldots + r^{n-1} + r^n)$$

If the sum of this series is denoted by $S$, then some algebraic manipulation shows that

$$S - rS = a - ar^{n+1}.$$

Applying the distributive property to the common factors and solving for $S$ shows that

$$S(1 - r) = a(1 - r^{n+1}),$$

so that

$$S = \frac{a(1 - r^{n+1})}{1 - r}.$$

Students develop the ability to see expressions such as $A_n = A_0\left(1+\frac{.15}{12}\right)^n$ as describing the total value of an investment at 15% interest, compounded monthly, for a number of compoundings, $n$. Moreover, they can interpret the following equation as a type of geometric series that would calculate the total value in an investment account at the end of one year if $100 is deposited at the beginning of each month (**MHM2, MHM4, and MHM7**):

$$A_1 + A_2 + \ldots + A_{12} = 100\left(1+\frac{.15}{12}\right)^1 + 100\left(1+\frac{.15}{12}\right)^2 + \ldots + 100\left(1+\frac{.15}{12}\right)^{12}$$

They apply the formula for geometric series to find this sum.

**Polynomial, Rational, and Radical Relationships**

**Perform arithmetic operations on polynomials.**

**M.3HS.15(*)**

Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction and multiplication; add, subtract and multiply polynomials. Instructional Note: Extend beyond the quadratic polynomials found in Mathematics II.

**Understand the relationship between zeros and factors of polynomials.**

**M.3HS.16 (*)**

Know and apply the Remainder Theorem: For a polynomial p(x) and a number a, the remainder on division by x – a is p(a), so p(a) = 0 if and only if (x – a) is a factor of p(x).

**M.3HS.17 (*)**

Identify zeros of polynomials when suitable factorizations are available and use the zeros to construct a rough graph of the function defined by the polynomial.

**Use polynomial identities to solve problems.**

**M.3HS.18 (^)**

Prove polynomial identities and use them to describe numerical relationships. For example, the polynomial identity (x2 + y2)2 = (x2 – y2)2 + (2xy)2 can be used to generate Pythagorean triples. Instructional Note: This cluster has many possibilities for optional enrichment, such as relating the example in M.A2HS.10 to the solution of the system u2 + v2 = 1, v = t(u+1), relating the Pascal triangle property of binomial coefficients to (x + y)n+1 = (x + y)(x + y)n, deriving explicit formulas for the coefficients, or proving the binomial theorem by induction.

**M.3HS.19 (+,^)**
Know and apply the Binomial Theorem for the expansion of (x + y)n in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle. Instructional Note: This cluster has many possibilities for optional enrichment, such as relating the example in M.A2HS.10 to the solution of the system u2 + v2 = 1, v = t(u+1), relating the Pascal triangle property of binomial coefficients to (x + y)n+1 = (x + y)(x + y)n, deriving explicit formulas for the coefficients, or proving the binomial theorem by induction.

**Rewrite rational expressions.**
**M.3HS.20 (*)**
Rewrite simple rational expressions in different forms; write a(x)/b(x) in the form q(x) + r(x)/b(x), where a(x), b(x), q(x), and r(x) are polynomials with the degree of r(x) less than the degree of b(x), using inspection, long division, or, for the more complicated examples, a computer algebra system. Instructional Note: The limitations on rational functions apply to the rational expressions.

**M.3HS.21 (+)**
Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply and divide rational expressions. Instructional Note: Requires the general division algorithm for polynomials.

Students in Mathematics III continue to develop their understanding of the set of polynomials as a system analogous to the set of integers that exhibits certain properties, and they explore the relationship between the factorization of polynomials and the roots of a polynomial (**M.3HS.15–M.3HS.17**). When a polynomial $p(x)$ is divided by $(x-a)$, $p(x)$ is written as $p(x) = q(x) \bullet (x-a) + r$, where $r$ is a constant. This can be done by inspection or by polynomial long division (**M.3HS.21**).

It follows that $p(a) = q(a) \bullet (a-a) + r = q(a) \bullet 0 + r = r$, so that $(x-a)$ is a factor of $p(x)$ if and only if $p(a) = 0$. This result is generally known as the Remainder Theorem (**M.3HS.16**) and provides an easy check to see if a polynomial has a given linear factor. This topic should not be reduced to "synthetic division," which limits the theorem to a method of carrying numbers between registers—something easily done by a computer—while obscuring the reasoning that makes the result evident. It is important to regard the Remainder Theorem as a theorem, not a technique (**MHM3**) [UA Progressions Documents 2013b, 7].

Students use the zeros of a polynomial to create a rough sketch of its graph and connect the results to their understanding of polynomials as functions (**M.3HS.17**). The notion that polynomials can be used to approximate other functions is important in advanced mathematics courses such as Calculus, and standard **M.3HS.17** is the first step in a progression that can lead students, as an extension topic, to constructing polynomial functions whose graphs pass through any specified set of points in the plane (UA Progressions Documents 2013b, 7).

Additionally, polynomials form a rich ground for mathematical explorations that reveal relationships in the system of integers (**M.3HS.18**). For example, students can explore the sequence of squares 1, 4, 9, 16, 25, 36, … and notice the differences between them—3, 5, 7, 9 , 11—are consecutive odd integers. This mystery is explained by the polynomial identity $(n+1)^2 - n^2 = 2n + 1$, which can be justified by using pictures (UA Progressions Documents 2013b, 6).

In Mathematics III, students explore rational functions as a system analogous to the rational numbers. They see rational functions as useful for describing many real-world situations—for instance, when rearranging the equation $d = rt$ to express the rate as a function of the time for a fixed distance $d_0$ and obtaining $r = \frac{d_0}{t}$. Now students see that any two polynomials can be divided in much the same way that numbers are (provided the divisor is not 0). Students first understand rational expressions as similar to other expressions in algebra, except that rational expressions have the special form $\frac{a(x)}{b(x)}$ for both $a(x)$ and $b(x)$ polynomials in $x$. Students should evaluate various rational expressions for many values of $x$, perhaps discovering that when the degree of $b(x)$ is larger than the degree of $a(x)$, the value of the expression gets smaller in absolute value as $|x|$ gets larger. Developing an understanding of the behavior of rational expressions in this way helps students to see these expressions as functions and sets the stage for working with simple rational functions in the Functions domain.

| **Example: The Juice-Can Equations** | **M.3HS.34** |
|---|---|
| If someone wanted to investigate the shape of a juice can of minimal surface area, the investigation could begin in the following way. If the volume $V_0$ is fixed, then the expression for the volume of the can is $V_0 = \pi r^2 h$, where $h$ is the height of the can and $r$ is the radius of the circular base. On the other hand, the surface area $S$ is given by the following formula: <br><br> $$S = 2\pi rh + 2\pi r^2$$ <br><br> This is because the two circular bases of the can contribute $2\pi r^2$ units of surface area, and the outside surface of the can contributes an area in the shape of a rectangle with length equal to the circumference of the base, $2\pi r$, and height equal to $h$. Since the volume is | |

fixed, $h$ can be found in terms of $r$: $h = \frac{V_0}{\pi r^2}$. Then this can be substituted into the equation for the surface area:

$$S = 2\pi r \bullet \frac{V_0}{\pi r^2} + 2\pi r^2$$
$$= \frac{2V_0}{r} + 2\pi r^2$$

This equation expresses the surface area $S$ as a (rational) function of $r$, which can then be analyzed. (Also refer to standard **M.3HS.39**)

Additionally, students are able to rewrite rational expressions in the form $a(x) = q(x) \bullet b(x) + r(x)$, where $r(x)$ is a polynomial of degree less than $b(x)$, by inspection or by using polynomial long division. They can flexibly rewrite this expression as $\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}$ as necessary—for example, to highlight the end behavior of the function defined by the expression $\frac{a(x)}{b(x)}$. In order to make working with rational expressions more than just an exercise in the proper manipulation of symbols, instruction should focus on the characteristics of rational functions that can be understood by rewriting them in the ways described above (e.g., rates of growth, approximation, roots, axis intersections, asymptotes, end behavior, and so forth).

**Mathematical Modeling**

**Create equations that describe numbers or relationships.**

**M.3HS.31 (*,^)**
Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. Instructional Note: Use all available types of functions to create such equations, including root functions, but constrain to simple cases.

**M.3HS.32 (*,^)**
Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. Instructional Note: While functions will often be linear, exponential or quadratic the types of problems should draw from more complex situations than those addressed in Mathematics I. For example, finding the equation of a line through a given point perpendicular to another line allows one to find the distance from a point to a line.

**M.3HS.33 (*,^)**
Represent constraints by equations or inequalities and by systems of equations and/or inequalities and interpret solutions as viable or non-viable options in a modeling context. (e.g., Represent inequalities describing nutritional and cost constraints on combinations of different foods.)

**M.3HS.34 (*,^)**
Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. (e.g., Rearrange Ohm's law V = IR to highlight resistance R.) Instructional Note: The example given applies to earlier instances of this standard, not to the current course.

Students in Mathematics III work with all available types of functions, including root functions, to create equations (**M.3HS.31**). Although functions referenced in standards **M.3HS.32– M.3HS.34** will often be linear, exponential, or quadratic, the types of problems should draw from more complex situations than those addressed in Mathematics I and Mathematics II. For example, knowing how to find the equation of a line through a given point perpendicular to another line allows one to find the distance from a point to a line. The Juice-Can Equation example presented previously in this section is connected to standard **M.3HS.34**.

**Reasoning with Equations and Inequalities**

**Understand solving equations as a process of reasoning and explain the reasoning.**

**M.3HS.22 (*)**
Solve simple rational and radical equations in one variable and give examples showing how extraneous solutions may arise. Instructional Note: Extend to simple rational and radical equations.

**Represent and solve equations and inequalities graphically.**

**M.3HS.23 (*,^)**
Explain why the x-coordinates of the points where the graphs of the equations y = f(x) and y = g(x) intersect are the solutions of the equation f(x) = g(x); find the solutions approximately (e.g., using technology to graph the functions, make tables of values or find successive approximations. Include cases where f(x) and/or g(x) are linear, polynomial, rational, absolute value, exponential and logarithmic functions. Instructional Note: Include combinations of linear, polynomial, rational, radical, absolute value, and exponential functions.

Students extend their equation-solving skills to those involving rational expressions and radical equations, and they make sense of extraneous solutions that may arise (**M.3HS.22**). In particular, students understand that when solving equations, the flow of reasoning is generally forward, in the sense that it is assumed a number $x$ is a solution of the equation and then a list of possibilities for $x$ is found. However, not all steps in this process are reversible. For example, although it is true that if $x = 2$, then $x^2 = 4$, it is not true that if $x^2 = 4$, then $x = 2$, as $x = -2$ also satisfies this equation (UA Progressions Documents 2013b, 10).

Thus students understand that some steps are reversible and some are not, and they anticipate extraneous solutions. Additionally, students continue to develop their understanding of solving equations as solving for values of $x$ such that $f(x) = g(x)$, now including combinations of linear, polynomial, rational, radical, absolute value, and exponential functions (**M.3HS.23**). Students also understand that some equations can be solved only approximately with the tools they possess.

## Conceptual Category: Geometry

In Mathematics III, students extend their understanding of the relationship between algebra and geometry as they explore the equations for circles and parabolas. They also expand their understanding of trigonometry to include finding unknown measurements in non-right triangles. The Geometry standards included in the Mathematics III course offer many rich opportunities for students to practice mathematical modeling.

| Trigonometry of General Triangles and Trigonometric Functions |
|---|
| **Apply trigonometry to general triangles.**<br>**M.3HS.25 (+,^)**<br>Derive the formula A = 1/2 ab sin(C) for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.<br><br>**M.3HS.26 (+, ^)**<br>Prove the Laws of Sines and Cosines and use them to solve problems. Instructional Note: With respect to the general case of the Laws of Sines and Cosines, the definitions of sine and cosine must be extended to obtuse angles.<br><br>**M.3HS.27 (+,^)**<br>Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems and/or resultant forces). |

Students advance their knowledge of right-triangle trigonometry by applying trigonometric ratios in non-right triangles. For instance, students see that by dropping an altitude in a given triangle, they divide the triangle into right triangles to which these relationships can be applied. By seeing that the base of the triangle is $a$ and the height is $b \bullet \sin C$, they derive a general formula for the area of any triangle $A = \frac{1}{2}ab\sin(C)$ (**M.3HS.25**). Additionally, students use reasoning about similarity and trigonometric identities to derive the Laws of Sines and Cosines only in acute triangles, and they use these and other relationships to solve problems (**M.3HS.26–M.3HS.27**). Instructors will need to address the ideas of the sine and cosine of angles larger than or equal to 90 degrees to discuss Laws of Sine and Cosine fully, although full unit-circle trigonometry need not be discussed in this course.

| Mathematical Modeling |
| --- |
| **Visualize relationships between two dimensional and three-dimensional objects.** |
| **M.3HS.45 (*,^)** |
| Identify the shapes of two-dimensional cross-sections of three dimensional objects and identify three-dimensional objects generated by rotations of two-dimensional objects. |
| **Apply geometric concepts in modeling situations.** |
| **M.3HS.46 (*,^)** |
| Use geometric shapes, their measures and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). |
| **M.3HS.47 (*,^)** |
| Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile or BTUs per cubic foot). |
| **M.3HS.48 (*,^)** |
| Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost and/or working with typographic grid systems based on ratios). |

These standards are rich with opportunities for students to apply modeling (**MHM4**) with geometric concepts—and although these standards appear later in the sequence of the Mathematics III geometry standards, they should be incorporated throughout the geometry curriculum of the course. Standard **M.3HS.46** calls for students to use geometric shapes, their measures, and their properties to describe objects. This standard can involve two- and three-dimensional shapes and is not relegated to simple applications of formulas. Standard **M.3HS.48** calls for students to solve design problems by modeling with geometry.

| Example: Ice-Cream Cone | M.3HS.48 |
|---|---|

The owner of a local ice-cream parlor has hired you to assist with his company’s new venture: the company will soon sell its ice-cream cones in the freezer section of local grocery stores. The manufacturing process requires that each ice-cream cone be wrapped in a cone-shaped paper wrapper with a flat, circular disc covering the top. The company wants to minimize the amount of paper that is wasted in the process of wrapping the cones. Use a real ice-cream cone or the dimensions of a real ice-cream cone to complete the following tasks:

a. Sketch a wrapper like the one described above, using the actual size of your cone. Ignore any overlap required for assembly.
b. Use your sketch to help develop an equation the owner can use to calculate the surface area of a wrapper (including the lid) for another cone, given that its base had a radius and a slant height.
c. Using measurements of the radius of the base and slant height of your cone, and your equation from step b, find the surface area of your cone.
d. The company has a large rectangular piece of paper that measures 100 centimeters by 150 centimeters. Estimate the maximum number of complete wrappers sized to fit your cone that could be cut from this single piece of paper, and explain your estimate. (Solutions can be found at https://www.illustrativemathematics.org/ [accessed April 1, 2015].)

*Source:* Illustrative Mathematics 2013l.

Students further their understanding of the connection between algebra and geometry by applying the definition of circles and parabolas to derive equations and then deciding whether a given quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$ represents a circle or a parabola.

## Conceptual Category: Statistics and Probability

In Mathematics III, students develop a more formal and precise understanding of statistical inference, which requires a deeper understanding of probability. They explore the conditions that meet *random sampling* of a population and that allow for generalization of results to that population. Students also learn to use significant differences to make inferences about data gathered during the course of experiments.

| Inferences and Conclusions from Data |
|---|
| **Summarize, represent, and interpret data on single count or measurement variable.**<br>**M.3HS.1(*)**<br>Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets and tables to estimate areas under the normal curve.  Instructional Note:  While students may have heard of the normal distribution, it is unlikely that they will have prior experience using it to make specific estimates. Build on students' understanding of data distributions to help them see how the normal distribution uses area to make estimates of frequencies (which can be expressed as probabilities). Emphasize that only some data are well described by a normal distribution. |

Although students in Mathematics III may have heard of the normal distribution, it is unlikely that they will have had prior experience using it to make specific estimates. In Mathematics III, students build on their understanding of data distributions to see how to use the area under the normal distribution to make estimates of frequencies (which can be expressed as probabilities). It is important for students to see that only some data are well described by a normal distribution (**M.3HS.1**). Additionally, they can learn through examples the *empirical rule*: that for a normally distributed data set, 68% of the data lie within one standard deviation of the mean and 95% are within two standard deviations of the mean.

**Example: The Empirical Rule** **M.3HS.1**

Suppose that SAT mathematics scores for a particular year are approximately normally distributed, with a mean of 510 and a standard deviation of 100.

a. What is the probability that a randomly selected score is greater than 610?
b. What is the probability that a randomly selected score is greater than 710?
c. What is the probability that a randomly selected score is between 410 and 710?
d. If a student's score is 750, what is the student's percentile score (the proportion of scores below 750)?

***Solutions:***

a. The score 610 is one standard deviation above the mean, so the tail area above that is about half of 0.32, or 0.16. The calculator gives 0.1586.
b. The score 710 is two standard deviations above the mean, so the tail area above that is about half of 0.05, or 0.025. The calculator gives 0.0227.
c. The area under a normal curve from one standard deviation below the mean to two standard deviations above the mean is about 0.815. The calculator gives 0.8186.

d. Using either the normal distribution given or the standard normal (for which 750 translates to a z-score of 2.4), the calculator gives 0.9918.

## Inferences and Conclusions from Data

### Understand and evaluate random processes underlying statistical experiments.

**M.3HS.2(*)**

Understand that statistics allows inferences to be made about population parameters based on a random sample from that population.

**M.3HS.3(*)**

Decide if a specified model is consistent with results from a given data-generating process, for example, using simulation. (e.g., A model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?) Instructional Note: Include comparing theoretical and empirical results to evaluate the effectiveness of a treatment.

### Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

**M.3HS.4 (*,^)**

Recognize the purposes of and differences among sample surveys, experiments and observational studies; explain how randomization relates to each. Instructional Note: In earlier grades, students are introduced to different ways of collecting data and use graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data is collected determines the scope and nature of the conclusions that can be drawn from that data. The concept of statistical significance is developed informally through simulation as meaning a result that is unlikely to have occurred solely as a result of random selection in sampling or random assignment in an experiment.

**M.3HS.5 (*,^)**

Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. Instructional Note: Focus on the variability of results from experiments—that is, focus on statistics as a way of dealing with, not eliminating, inherent randomness.

**M.3HS.6 (*,^)**

Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. Instructional Note: Focus on the variability of results from experiments—that is, focus on statistics as a way of dealing with, not eliminating, inherent randomness.

**M.3HS.7 (*,^)**
Evaluate reports based on data. Instructional Note: In earlier grades, students are introduced to different ways of collecting data and use graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data is collected determines the scope and nature of the conclusions that can be drawn from that data. The concept of statistical significance is developed informally through simulation as meaning a result that is unlikely to have occurred solely as a result of random selection in sampling or random assignment in an experiment.

Students in Mathematics III move beyond analysis of data to make sound statistical decisions based on probability models. The reasoning process is as follows: develop a statistical question in the form of a hypothesis (supposition) about a population parameter, choose a probability model for collecting data relevant to that parameter, collect data, and compare the results seen in the data with what is expected under the hypothesis. If the observed results are far from what is expected and have a low probability of occurring under the hypothesis, then that hypothesis is called into question. In other words, the evidence against the hypothesis is weighed by probability (**M.3HS.2**) [UA Progressions Documents 2012d]. By investigating simple examples of simulations of experiments and observing outcomes of the data, students gain an understanding of what it means for a model to fit a particular data set (**M.3HS.3**). This includes comparing theoretical and empirical results to evaluate the effectiveness of a treatment.

In earlier grade levels, students are introduced to different ways of collecting data and use graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data are collected determines the scope and nature of the conclusions that can be drawn from those data. The concept of *statistical significance* is developed informally through simulation as meaning a result that is unlikely to have occurred solely through random selection in sampling or random assignment in an experiment. When covering standards **M.3HS.5– M.3HS.6**, instructors should focus on the variability of results from experiments—that is, on statistics as a way of handling, not eliminating, inherent randomness. Because standards **M.3HS.2– M.3HS.7** are all modeling standards, students should have ample opportunities to explore statistical experiments and informally arrive at statistical techniques.

| **Example: Estimating a Population Proportion** | **M.3HS.5** |
|---|---|
| Suppose a student wishes to investigate whether 50% of homeowners in her neighborhood will support a new tax to fund local schools. If she takes a random sample of 50 homeowners in her neighborhood, and 20 support the new tax, then the *sample proportion* | |

agreeing to pay the tax would be 0.4. But is this an accurate measure of the true proportion of homeowners who favor the tax? How can this be determined?

If this sampling situation (**MHM4**) is simulated with a graphing calculator or spreadsheet software under the assumption that the true proportion is 50%, then the student can arrive at an understanding of the *probability* that her randomly sampled proportion would be 0.4. A simulation of 200 trials might show that 0.4 arose 25 out of 200 times, or with a probability of 0.125. Thus, the chance of obtaining 40% as a sample proportion is not insignificant, meaning that a true proportion of 50% is plausible.

Adapted from UA Progressions Documents 2012d.

**Inferences and Conclusions from Data**

**Use probability to evaluate outcomes of decisions.**

**M.3HS.8 (+, ^)**
Use probabilities to make fair decisions (e.g., drawing by lots or using a random number generator).

**M.3HS.9 (+, ^)**
Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, and/or pulling a hockey goalie at the end of a game). Instructional Note: Extend to more complex probability models. Include situations such as those involving quality control or diagnostic tests that yield both false positive and false negative results.

As in Mathematics II, students apply probability models to make and analyze decisions. This skill is extended in Mathematics III to more complex probability models, including situations such as those involving quality control or diagnostic tests that yield both false-positive and false-negative results. See the University of Arizona Progressions document titled "High School Statistics and Probability" for further explanation and examples: http://ime.math.arizona.edu/progressions/ (UA Progressions Documents 2012d [accessed April 6, 2015]).

Mathematics III is the culmination of the three-course sequence in the Integrated Pathway for mathematics. Students completing this pathway will be well prepared for advanced mathematics and should be encouraged to continue their study of mathematics with Mathematics IV or other mathematics courses from the Fourth Course Options listed in the West Virginia College- and Career Readiness Standards, dual credit mathematics courses or courses offered through WV Virtual School. Policy 2510 requires that, beginning with the 2018-2019 freshman cohort, all students must be enrolled in a mathematics course during each year of high school.

## Common Misconceptions – By Domain

### Number and Quantity

- Students often have difficulty differentiating between finding the solution to an equation such as $x^2 = 25$ and taking the square root of a number. The equation $x^2 = 25$ has two solutions, 5 and -5. Expressions such as $\sqrt{3^2}$ or $\sqrt{(-7)^2}$ ask for the "principal" square root. As a result, $\sqrt{3^2}$ = 3 and $\sqrt{(-7)^2}$ = 7

- Student confusion in differentiating between expressions involving the square of a negative number and the negative of a square, often extend to square roots. Students may need help in appreciating the Order of Operations in differentiating between the expressions.

$$(-5)^2 \rightarrow (-5)(-5) \rightarrow 25$$
$$-(5)^2 \rightarrow -(5)(5) \rightarrow -25$$
$$\sqrt{(-5)^2} \rightarrow \sqrt{25} \rightarrow 5$$
$$-\sqrt{5^2} \rightarrow -\sqrt{25} \rightarrow -5$$
$$\sqrt{(5)^2} \rightarrow \sqrt{-25} \rightarrow 5$$

- Students may have difficulty when negative exponents involve fractions. Students may incorrectly believe that $16^{-\frac{1}{2}}$ means -4 rather than $\frac{1}{4}$.

$$16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$$

### Algebra

- Students may have difficulty distinguishing between the terms of an expression and the variable of an expression. For example, students may reason that the trinomial $3a + b + 5cd$ has four terms because the student is counting variable rather than terms. Often students do not recognize a constant as a term and incorrectly identify the trinomial $4x - 2y + 7$ as having two terms.

- Students who rely solely on procedures may believe they need multiply the binomials in an equation such as $(x + 4)(x - 3) = 0$ and then factor the resulting expression to solve for the zeros. Students need to develop an understanding of the concept of the Zero Factor Property. Students may experience similar confusion in applying the Zero Factor Property in solving equations such as $5(x - 7)(x + 1) = 0$ or $6x(x - 7) = 0$.

- Students who mistakenly believe that $\sqrt{x^2 + y^2}$ is equivalent to $x + y$, may deduce this from the misconception that $(x + y)^2$ is equivalent to $x^2 + y^2$.

- Students who mistakenly equate $(x + y)^2$ with $x^2 + y^2$ or equate $(x - y)^2$ with $x^2 - y^2$ have misconceptions regarding the concept of finding the square of a number or expression. These misconceptions may stem from students' prior difficulty in recognizing that $(-5)^2$ → (-5)(-5) →25 or that $-(5)^2$ → –(5)(5) → -25.

- Students who incorrectly equate rational expressions such as $\frac{x^2 - 6x + 9}{x - 3}$ with $x - 2x + 9$ or $x - 6x - 3$ or $x^2 + 2x + 9$, etc., have difficulty distinguishing between terms and factors.

- Students may demonstrate difficulties in solving the equation $\sqrt{x - 1}$ = $x - 7$. Students may appropriately decide to square both sides of the equation and write $(\sqrt{x - 1})^2$ = $(x - 7)^2$ Student error may arise in writing $(\sqrt{x - 1})^2$ = $x^2$ - $7^2$, demonstrating misconceptions in squaring binomials (or multiplying) binomials.

- The procedure for solving equations and inequalities are so similar. As a result, students often forget to attend to precision when multiplying and dividing by a negative number when solving inequalities. Students may correctly solve equations such as $-2x = 6$ and find its solution to be $x = -3$. Misconceptions arise when students equate solving the equation with a similar inequality $-2x \leq 6$ and incorrectly determine that $x \leq -3$. The misconception can be addressed by encouraging students to verify solutions to equations and to inequalities.

- Students often solve rational and radical equations without checking to determine if any solution may be erroneous. Students should be encouraged to verify solutions.

## Functions

- Students may erroneously interpret the notation $g(3)$ to mean "$g\ times\ 3$".

- Students often believe that all functions must use the symbols $f, x$, and $y$.

- When graphing students may confuse the parts of the slope-equation form of a linear equation. Students may incorrectly determine that the function $g(x) = 2x + \frac{3}{5}$ has a slope of $\frac{3}{5}$ and a y-intercept of 2. Students may be relying on a procedural understanding that the y-intercept is always an integer and the slope or $\frac{rise}{run}$ must be a fraction.

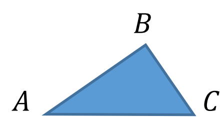
- Students often incorrectly assume that the function $f(x + k)$ where $k > 0$ will result in a horizontal shift of the graph $k$ units to the right.

## Geometry

- Students often think of congruence as "figures with the same shape and size." While this understanding is not incorrect, it is important to that the continually emphasize the link of congruence with rigid motions and show that rigid motions do in fact produce "figures with the same shape and size."

- Students may look at scale factor as the distance that is added on to the original distance. Dilations where the center of dilation is a vertex of a figure can prove challenging because the sides of the pre-image and image overlap.

- Students may have difficulty identifying the relevant sides in relationship to a given angle, especially If the triangle is not depicted in a typical position where its legs are horizontal and vertical. For example, given $\Delta\, ABC$ below with right angle $B$, students may struggle to understand that $\sin A = \frac{BC}{AC}$ and that $\sin A \neq \frac{BC}{AB}$.




- Students may have difficulty differentiating between the meaning of the statements $\Delta\, ABC \cong \Delta\, DEF$ and $\Delta\, ABC \sim \Delta\, DEF$. While $\Delta\, ABC \cong \Delta\, DEF$ implies that AB = DE, $\Delta\, ABC \sim \Delta\, DEF$ does not imply that AB = DE.

- Students often have difficulty differentiating among $\overleftarrow{EF}$, $\overrightarrow{EF}$, $\overleftrightarrow{EF}$, and $\overline{EF}$.

## Statistics and Probability

- Students often use the word *outlier* inaccurately, failing to verify that it satisfies the necessary conditions. Using terms such as *unusual feature* or *data point* can help students avoid using the term outlier when it is not appropriate.

- Students commonly believe that any data that is collected should follow a normal distribution.

- Students need to understand that association does not necessarily provide evidence for cause and effect.

Steven L. Paine, Ed.D.
West Virginia Superintendent of Schools