West Virginia math4life

Educators' Guide for Mathematics

Mathematics IV
Trigonometry/Pre-Calculus

West Virginia DEPARTMENT OF EDUCATION

Mathematics IV-Trigonometry/Pre-calculus

Mathematics IV-Trigonometry/Pre-calculus combines concepts of trigonometry, geometry and algebra that are needed to prepare students for the study of calculus. The course strengthens students' conceptual understanding of problems and mathematical reasoning in solving problems. Familiarity with these topics is especially important for students who intend to study calculus, physics, other sciences, and engineering in college. The main topics in the Mathematics IV-Trigonometry/Pre-calculus course are complex numbers, rational functions, and vectors and matrices. Students who enroll in Mathematics IV-Trigonometry/Pre-calculus should have met the West Virginia College- and Career-Readiness standards in the integrated or traditional pathway. It is recommended that students complete Math IV-Trigonometry/Pre-calculus before taking an AP® calculus course.

What Students Learn in Mathematics IV-Trigonometry/Pre-calculus

Students in Mathematics IV-Trigonometry/Pre-calculus extend their work with complex numbers, which started in Mathematics III or Algebra II, to see that complex numbers can be represented in the Cartesian plane and that operations with complex numbers have a geometric interpretation.

Students begin working with vectors, representing them geometrically and performing operations with them. They connect the notion of vectors to complex numbers. Students also work with matrices and their operations, experiencing for the first time an algebraic system in which multiplication is not commutative. Additionally, they see the connection between matrices and transformations of the plane—namely, that a vector in the plane can be multiplied by a 2 x 2 matrix to produce another vector—and they work with matrices from the perspective of transformations. They also find inverse matrices and use matrices to represent and solve linear systems.

Students extend their work with trigonometric functions, investigating the reciprocal functions *secant*, *cosecant*, and *cotangent* and the graphs and properties associated with those functions. Students find inverse trigonometric functions by appropriately restricting the domains of the standard trigonometric functions and use them to solve problems that arise in modeling contexts.

Although students in Mathematics IV-Trigonometry/Pre-calculus have worked previously with parabolas and circles, they now work with ellipses and hyperbolas.

Finally, students work with rational functions that are more complicated, graphing them and determining zeros, y-intercepts, symmetry, asymptotes, intervals for which the function is increasing or decreasing, and maximum or minimum points.

Connecting Mathematical Habits of Mind

The Mathematical Habits of Mind (**MHM**) apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a rigorous, coherent, useful, and logical subject. The Mathematical Habits of Mind represent a picture of what it looks like for students to *do mathematics* and, to the extent possible, content instruction should include attention to appropriate practice standards. The Mathematics IV-Trigonometry/Pre-calculus course presents examples of how students can engage with each Mathematical Habit of Mind. The following table offers some examples.

Mathematical Habits of Mind—Explanation and Examples for Mathematics IV-Trigonometry/Pre-calculus

Mathematical Habits of Mind	Explanation and Examples
MHM1 Make sense of problems andpersevere in solving them.	Students expand their repertoire of expressions and functions that can be used to solve problems. They grapple with understanding the connection between complex numbers and vectors.
MHM2 Reason abstractly andquantitatively.	Students understand the connection between transformations and matrices, seeing a matrix as an algebraic representation of a transformation of the plane.
MHM3 Construct viable arguments and critique the reasoning of others.	Students continue to reason through the solution of an equation and justify their reasoning to their peers. They defend their choice of a function to model a real-world situation.
MHM4 Model with mathematics.	Students apply their new mathematical understanding to real-world problems. They also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
MHM5 Use appropriate tools strategically.	Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions.
MHM6 Attend to precision.	Students make note of the precise definition of *complex number*, understanding that real numbers are a subset of complex numbers. They pay attention to units in real-world problems and use unit analysis as a method for verifying answers.
MHM7 Look for and make use ofstructure.	Students understand that matrices form an algebraic system in which the order of multiplication matters, especially when solving linear systems using matrices.
MHM8 Look for and express regularity inrepeated reasoning.	Students multiply several vectors by matrices and observe that some matrices produce rotations or reflections. They compute with complex numbers and generalize the results to understand the geometric nature of their operations.

MHM4 holds a special place throughout the higher mathematics curriculum, as modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a high place in instruction.

Mathematics IV-Trigonometry/Pre-calculus Content Standards

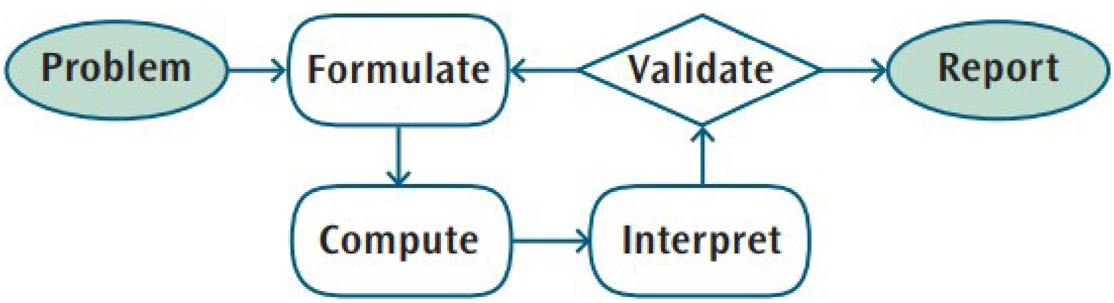
The Mathematics IV-Trigonometry/Pre-calculus course is organized by domains, clusters, and standards. The overall purpose and progression of the standards included in the Mathematics IV-Trigonometry/Pre-calculus course are described below, according to each conceptual category. Note that the standards are not listed in an order in which they should be taught.

Conceptual Category: Modeling

Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them, and the mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise: *Which of the quantities present in this situation are known and which are unknown? What can I generalize? Is there some way to introduce into this diagram a known shape that gives more information?* Students need to decide on a solution path, which may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They try to use previously derived models (e.g., linear functions), but may find that a new equation or function will apply. Additionally, students may see when trying to answer their question that solving an equation arises as a necessity and that the equation often involves the specific instance of knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, and the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see the diagram below of a model cycle. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Modeling Cycle



The examples for the course are framed as much as possible to illustrate the concept of mathematical modeling.

Conceptual Category: Functions

The standards of the Functions conceptual category can set the stage for students to learn other standards in Mathematics IV-Trigonometry/Precalculus. At this level, expressions are often viewed as defining outputs for functions, and algebraic manipulations are then performed meaningfully with an eye toward what can be revealed about the function.

Analysis and Synthesis of Functions
Analyze functions using different representations. **M.4HSTP.19** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. Instructional Note: This is an extension of graphical analysis from Math III or Algebra II that develops the key features of graphs with the exception of asymptotes. Students examine vertical, horizontal, and oblique asymptotes by considering limits. Students should note the case when the numerator and denominator of a rational function share a common factor. Utilize an informal notion of limit to analyze asymptotes and continuity in rational functions. Although the notion of limit is developed informally, proper notation should be followed.
Build a function that models the relationship between two quantities. **M.4HSTP.20** Write a function that describes a relationship between two quantities, including composition of functions. For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of the water balloon as a function of time, then $T(h(t)$ is the temperature at the location of the weather balloon as a function of time.
Build new functions from existing functions. **M.4HSTP.21** Find inverse functions. Instructional Note: This is an extension of concepts from Mathematics III where the idea of inverse functions was introduced. a. Verify by composition that one function is the inverse of another. b. Read values of an inverse function from a graph or a table, given that the function has an inverse. (Instructional Note: Students must realize that inverses created through function composition produce the same graph as reflection about the line y=x.) c. Produce an invertible function from a non-invertible function by restricting the domain. (Instructional Note: Systematic procedures must be developed for restricting domains of non-invertible functions so that their inverses exist.) **M.4HSTP.22** Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Students further their understanding of inverse functions. Previously, students found inverse functions only in simple cases (e.g., solving for x, when $f(x) = c$, finding inverses of linear functions); in Mathematics IV-Trigonometry/Pre-calculus they explore the relationship between two functions that are inverses of each other (i.e., that f and g are inverses if $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$). They may also begin to use inverse function notation, expressing as $g = f^{-1}$ (**MHM2, MHM6**). Students in Mathematics IV-Trigonometry/Precalculus construct inverse functions by appropriately restricting the domain of a given function and use inverses in different contexts. They understand how a function and its domain and range are related to its inverse function. They realize that finding an inverse function is more than simply "switching variables" and solving an equation. They can even find simpler inverses mentally, such as when they reverse the "steps" for the equation $f(x) = x^3 - 1$ to realize that the inverse of f must be $f^{-1}(x) = \sqrt[3]{x+1}$ (**MHM7**).

Trigonometric and Inverse Trigonometric Functions of Real Numbers

Extend the domain of trigonometric functions using the unit circle.

M.4HSTP.23

Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number. Instructional Note: Students use the extension of the domain of trigonometric functions developed in Math III to obtain additional special angles and more general properties of the trigonometric functions.

M.4HSTP.24

Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions.

M.4HSTP.25

Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

M.4HSTP.26

Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology and interpret them in terms of the context. Instructional Note: Students should draw analogies to the work with inverses in the previous unit.

M.4HSTP.27

Solve more general trigonometric equations (e.g., $2sin^2x + \sin x - 1 = 0$ can be solved using factoring).

Prove and apply trigonometric identities.

M.4HSTP.28

Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Apply transformations of function to trigonometric functions.

M.4HSTP.29

Graph trigonometric functions showing key features, including phase shift. Instructional Note: In Math III, students graphed trigonometric functions showing period, amplitude, and vertical shifts.)

These standards call for students to expand their understanding of the trigonometric functions by connecting properties of the functions to the unit circle. For example, students understand that since traveling 2π radians around the unit circle returns one to the same point on the circle, this must be reflected in the graphs of sine and cosine (**MHM8**). Students extend their knowledge of finding inverses to trigonometric functions and use these inverses in a wide range of application problems.

Students in Mathematics IV-Trigonometry/Precalculus derive the addition and subtraction formulas for sine, cosine, and tangent. Another opportunity for connections arises here, as students can investigate the relationship between these formulas and complex multiplication.

Conceptual Category: Number and Quantity

The Number and Quantity standards in Mathematics IV-Trigonometry/Precalculus represent a culmination of students' understanding of number systems. Students investigate the geometry of complex numbers more fully and connect it to operations with complex numbers. Additionally, students develop the notion of a vector and connect operations with vectors and matrices to transformations of the plane.

Building Relationships among Complex Numbers, Vectors, and Matrices

Perform arithmetic operations with complex numbers.

M.4HSTP.1
Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers. Instructional Note: In Math II students extended the number system to include complex numbers and performed the operations of addition, subtraction, and multiplication.

Represent complex numbers and their operations on the complex plane.

M.4HSTP.2
Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

M.4HSTP.3
Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation (e.g., $(-1 + \sqrt{3}i)^3 = 8$ because $(-1 + \sqrt{3}i)$ has modulus 2 and argument 120°).

M.4HSTP.4
Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Represent and model with vector quantities.

M.4HSTP. 5
Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., v, $|v|$, $||v||$). Instructional Note: This is the student's first experience with vectors. The vectors must be represented both geometrically and in component form with emphasis on vocabulary and symbols.

M.4HSTP.6
Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

M.4HSTP.7
Solve problems involving velocity and other quantities that can be represented by vectors.

Building Relationships among Complex Numbers, Vectors, and Matrices

Perform operations on vectors.

M.4HSTP.8

Add and subtract vectors.

a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
c. Understand vector subtraction $v - w$ as $v + (-w)$, where $-w$ is the additive inverse of w, with the same magnitude as w and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

M.4HSTP.9

Multiply a vector by a scalar.

a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(vx, vy) = (cvx, cvy)$.
b. Compute the magnitude of a scalar multiple cv using $\|cv\| = |c| \cdot \|v\|$. Compute the direction of cv knowing that when $|c|v \neq 0$, the direction of cv is either along v (for $c > 0$) or against v (for $c < 0$).

Perform operations on matrices and use matrices in applications.

M.4HSTP.10

Use matrices to represent and manipulate data (e.g., to represent payoffs or incidence relationships in a network.)

M.4HSTP.11

Multiply matrices by scalars to produce new matrices (e.g., as when all of the payoffs in a game are doubled.)

M.4HSTP.12

Add, subtract, and multiply matrices of appropriate dimensions.

M.4HSTP.13

Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties. Instructional Note: This is an opportunity to view the algebraic field properties in a more generic context, particularly noting that matrix multiplication is not commutative.

M.4HSTP.14

Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is non-zero if and only if the matrix has a multiplicative inverse.

M.4HSTP.15

Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

M.4HSTP.16

Work with 2 x 2 matrices as transformations of the plane and interpret the absolute value of the determinant in terms of area. Instructional Note: Matrix multiplication of a 2 x 2 matrix by a vector can be interpreted as transforming points or regions in the plane to different points or regions. In particular a matrix whose determinant is 1 or -1 does not change the area of a region.

As mentioned previously, complex numbers and vectors should be taught with an emphasis on connections between them. For instance, students connect the addition of complex numbers to the addition of vectors.

Students investigate vectors as geometric objects in the plane that can be represented by ordered pairs, and they investigate matrices as objects that act on vectors. By working with vectors and matrices both geometrically and quantitatively, students discover that vector addition and subtraction behave according to certain properties, while matrices and matrix operations observe their own set of rules. Attending to structure, students discover with matrices a new set of mathematical objects and operations involving multiplication that is not commutative. They find inverse matrices by hand in 2 x 2 cases and use technology in other cases. Work with vectors and matrices sets the stage for solving systems of equations in the Algebra conceptual category.

Conceptual Category: Algebra

In the Algebra conceptual category, Mathematics IV-Trigonometry/Precalculus students work with higher-degree polynomials and rational functions that are more complicated. As always, they attend to the meaning of the expressions they work with, and the expressions they encounter often arise in the context of functions. As in all other higher mathematics courses, students in Mathematics IV-Trigonometry/Precalculus work with creating and solving equations and do so in contexts connected to real-world situations through modeling.

By the time students take Mathematics IV-Trigonometry/Precalculus, they should have a well-developed understanding of the concept of a function. To make work with rational expressions more meaningful, students should be given opportunities to connect rational expressions to rational *functions* (whose outputs are defined by the ex-pressions). For example, a traditional exercise with rational expressions might have the following form:

$$\text{Simplify } \frac{200}{x} = \frac{100}{x-10}$$

The intention here is that students will find a common denominator and transform the expression into $\frac{300x-2000}{x(x-10)}$. In contrast, students could view the two expressions as defining the outputs of two functions—f and g, respectively—where $f(x) = \frac{200}{x}$ and $g(x) = \frac{100}{x-10}$ (**MHM2**). In this case, f could be the function that represents the time it takes for a car to travel 200 miles at an average speed of x miles per hour, and g could be the function that represents the time it takes for the car to travel 100 miles at an average speed that is 10 miles per hour slower (**MHM4**). Students can be asked to consider the domains of the two functions, the domain on which the sum of the two functions defined by $(f+g)(x) = f(x) + g(x)$ makes sense, and what the sum denotes (total time to travel the 300 miles altogether). Furthermore, students can calculate tables of outputs for the two functions using a spreadsheet, add the outputs on the spreadsheet, and then graph the resulting outputs, discovering that the data fit the graph of the equation $y = \frac{300x-2000}{x(x-10)}$. Finally, if these expressions arise in a modeling context, students can interpret the results of studying these functions and their sum in the real-world context.

Building Relationships among Complex Numbers, Vectors, and Matrices
Solve systems of equations. **M.4HSTP.17** Represent a system of linear equations as a single matrix equation in a vector variable. **M.4HSTP.18** Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3 x 3 or greater). Instructional Note: Students have earlier solved two linear equations in two variables by algebraic methods.

Students use matrix multiplication to connect their newfound knowledge of matrices to the representation of systems of linear equations. They can do this in modeling situations (e.g., those involving economic quantities or geometric elements).

Conceptual Category: Geometry

The standards of the Geometry conceptual category also connect to several other standards in the Mathematics IV-Trigonometry/Pre-calculus curriculum. For example, students continue to work with conic sections (started in Mathematics III or Algebra II).

Derivations in Analytic Geometry
Translate between the geometric description and the equation for a conic section. **M.4HSTP.30** Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant. Instructional Note: In Math II students derived the equations of circles and parabolas. These derivations provide meaning to the otherwise arbitrary constants in the formulas. **Explain volume formulas and use them to solve problems.** **M.4HSTP.31** Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures. Instructional Note: Students were introduced to Cavalieri's principle in Math II.

Students in Mathematics IV-Trigonometry/Precalculus continue to study trigonometric functions by discovering that these functions can also be used with general (non-right) triangles through the use of appropriate auxiliary lines. Students can then use these laws to solve problems, and they connect the relationships described by the laws to the geometry of vectors.

Modeling with Probability

Calculate expected values and use them to solve problems.

M.4HSTP.32

Define a random variable for a quality of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions. Instructional Note: Although students are building on their previous experience with probability in middle grades and in Math II and III, this is their first experience with expected value and probability distributions.

M.4HSTP.33

Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.

M.4HSTP.34

Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. (e.g., Find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.)

M.4HSTP.35

Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households? Instructional Note: It is important that students can interpret the probability of an outcome as the area under a region of a probability distribution graph.

Use probability to evaluate outcomes of decisions.

M.4HSTP.36

Weight the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

a. Find the expected payoff for a game of chance. (e.g., Find the expected winnings from a state lottery ticket or a game at a fast food restaurant.)
b. Evaluate and compare strategies on the basis of expected values. (e.g., Compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having minor or a major accident.)

Series and Informal Limits

Use sigma notations to evaluate finite sums.

M.4HSTP.37

Develop sigma notation and use it to write series in equivalent form. For example, write $\sum_{i=1}^{n}(3i^2+7)$ as $\sum_{i=1}^{n}i^2+7\sum_{i=1}^{n}1$.

M.4HSTP.38

Apply the method of mathematical induction to prove summation formulas. For example, verify that $\sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6}$. Instructional Note: Some students may have encountered induction in Math III in proving the Binomial Expansion Theorem, but for many this is their first experience.

Extend geometric series to infinite geometric series.

M.4HSTP.39

Develop intuitively that the sum of an infinite series of positive numbers can converge and derive the formula for the sum of an infinite geometric series. Instructional Note: In Math I, students described geometric sequences with explicit formulas. Finite geometric series were developed in Math III.

M.4HSTP.40

Apply infinite geometric series models. For example, find the area bounded by a Koch curve. Instructional Note: Rely on the intuitive concept of limit developed in unit 2 to justify that a geometric series converges if and only if the ratio is between -1 and 1.

Common Misconceptions – By Domain

Number and Quantity

- Students often have difficulty differentiating between finding the solution to an equation such as $x^2 = 25$ and taking the square root of a number. The equation $x^2 = 25$ two solutions, 5 and -5. Expressions such as $\sqrt{3^2}$ or $\sqrt{(-7)^2}$ ask for the "principal" square root. As a result, $\sqrt{3^2} = 3$ and $\sqrt{(-7)^2} = 7$.

- Student confusion in differentiating between expressions involving the square of a negative number and the negative of a square, often extend to square roots. Studentsmay need help in appreciating the Order of Operations in differentiating between the expressions.

$$(-5)^2 \rightarrow (-5)(-5) \rightarrow 25$$
$$-(5)^2 \rightarrow -(5)(5) \rightarrow -25$$
$$\sqrt{(-5)^2} \rightarrow \sqrt{25} \rightarrow 5$$
$$-\sqrt{5^2} \rightarrow -\sqrt{25} \rightarrow -5$$
$$\sqrt{(5)^2} \rightarrow \sqrt{25} \rightarrow 5$$

- Students may have difficulty when negative exponents involve fractions. Students may incorrectly believe that $16^{-\frac{1}{2}}$ means -4 rather than $\frac{3}{5}$

$$16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$$

Algebra

- Students may have difficulty distinguishing between the terms of an expression and the variable of an expression. For example, students may reason that the trinomial $3a + b + 5cd$ has four terms because the student is counting variable rather than terms. Often students do not recognize a constant as a term and incorrectly identify the trinomial $4x - 2y + 7$ as having two terms.

- Students who rely solely on procedures may believe they need multiply the binomials in an equation such as $(x + 4)(x - 3) = 0$ and then factor the resulting expression to solve for the zeros. Students need to develop an understanding of the concept of the Zero Factor Property. Students may experience similar confusion in applying the Zero Factor Property in solving equations such as $5(x - 7)(x + 1) = 0$ or $6x(x - 7) = 0$.

- Students who mistakenly believe that $\sqrt{x^2 + y^2}$ is equivalent to $x + y$, may deduce this from the misconception that $(x + y)^2$ is equivalent to $x^2 + y^2$.

- Students who mistakenly equate $(x + y)^2$ with $x^2 + y^2$ or equate $(x - y)^2$ with $x^2 - y^2$ have misconceptions regarding the concept of finding the square of a number or expression. These misconceptions may stem from students' prior difficulty in recognizing that $(-5)^2 \rightarrow (-5)(-5) \rightarrow 25$ or that $-(5)^2 \rightarrow -(5)(5) \rightarrow -25$.

- Students who incorrectly equate rational expressions such as $\frac{x^2-6x+9}{x-3}$ with x - $2x + 9$ or x - $6x - 3$ or $x^2 + 2x + 9$, etc., have difficulty distinguishing between terms and factors.

- Students may demonstrate difficulties in solving the equation $\sqrt{x-1=x-7}$. Students may appropriately decide to square both sides of the equation and write $(\sqrt{x-1})^2 = (x-7)^2$. Student error may arise in writing $(\sqrt{x-1})^2 = x^2 - 7^2$, demonstrating misconceptions in squaring binomials (or multiplying) binomials.

- The procedure for solving equations and inequalities are so similar. As a result, students often forget to attend to precision when multiplying and dividing by a negative number when solving inequalities. Students may correctly solve equations such as $-2x = 6$ and find its solution to be $x = -3$. Misconceptions arise when students equate solving the equation with a similar inequality $-2x \leq 6$ and incorrectly determine that $x \leq -3$. The misconception can be addressed by encouraging students to verify solutions to equations and to inequalities.

- Students often solve rational and radical equations without checking to determine if any solution may be erroneous. Students should be encouraged to verify solutions.

Functions

- Students may erroneously interpret the notation $g(3)$ to mean "g *times 3*".

- Students often believe that all functions must use the symbols f, x, and y.

- When graphing students may confuse the parts of the slope-equation form of a linear equation. Students may incorrectly determine that the function has a $g = 2x + \frac{3}{5}$ has a slope of $\frac{3}{5}$ and a y-intercept of 2. Students may be relying on a procedural understanding that the y-intercept is always an integer and the slope or $\frac{rise}{run}$ must be a fraction.

- Students often incorrectly assume that the function $f(x + k)$ where $k > 0$ will result in a horizontal shift of the graph k units to the right.

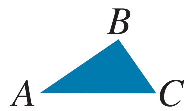
Geometry

- Students often think of congruence as "figures with the same shape and size." While this understanding is not incorrect, it is important to that the continually emphasize the link of congruence with rigid motions and show that rigid motions do in fact produce "figures with the same shape and size."

- Students may look at scale factor as the distance that is added on to the original distance. Dilations where the center of dilation is a vertex of a figure can prove challenging because the sides of the pre-image and image overlap.

- Students may have difficulty identifying the relevant sides in relationship to a given angle, especially if the triangle is not depicted in a typical position where its legs are horizontal and vertical. For example, given ΔABC below with right angle B, students may struggle to understand that $\sin A = \frac{BC}{AC}$ and that $A \neq \frac{BC}{AB}$.

- Students may have difficulty differentiating between the meaning of the statements $\Delta ABC \cong \Delta DEF$ and $\Delta ABC \sim \Delta DEF$. While $\Delta ABC \cong \Delta DEF$ implies that AB = DE, $\Delta ABC \sim \Delta DEF$ does not imply that AB = DE.



- Students often have difficulty differentiating among $\overleftarrow{EF}$, $\overrightarrow{EF}$, $\overleftrightarrow{EF}$, and $\overline{EF}$.

Statistics and Probability

- Students often use the word *outlier* inaccurately, failing to verify that it satisfies the necessary conditions. Using terms such as *unusual feature* or *data point* can help students avoid using the term outlier when it is not appropriate.

- Students commonly believe that any data that is collected should follow a normal distribution.

- Students need to understand that association does not necessarily provide evidence for cause and effect.

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