West Virginia math4life

Educators' Guide for Mathematics

Grade 6

West Virginia DEPARTMENT OF EDUCATION

Grade Six

Mathematics Instruction

Students in grade six build on a strong foundation to prepare for higher mathematics. Grade six is an especially important transition year bridging the concrete concepts of arithmetic and the abstract thinking of algebra (Arizona Department of Education [ADE] 2010). In grade six, instructional time should focus on four critical areas: (1) connecting ratio, rate, and percentage to whole number multiplication and division and using concepts of ratio and rate to solve problems; (2) completing understanding of division of fractions and extending the concept of number to the system of rational numbers, which includes negative numbers; (3) writing, interpreting, and using expressions and equations; and (4) developing understanding of statistical thinking (National Governors Association Center for Best Practices, Council of Chief State School Officers 2010m). Students also work toward fluency with multi-digit division and multi-digit decimal operations.

In previous grades, students build a foundation in number and operations, geometry, and measurement and data. When students enter grade six, they are fluent in addition, subtraction, and multiplication with multi-digit whole numbers and have a solid conceptual understanding of all four operations with positive rational numbers, including fractions. Students at this grade level have begun to understand measurement concepts (e.g., length, area, volume, and angles), and their knowledge of how to represent and interpret data is emerging (adapted from Charles A. Dana Center 2012).

West Virginia College- and Career-Readiness Standards for Mathematics

The West Virginia College- and Career-Readiness Standards for Mathematics (Policy 2520.2B) emphasize key content, skills, and practices at each grade level.

- Instruction is focused on grade-level standards.
- Instructions should be attentive to learning across grades and to linking major topics within grades.
- Instruction should develop conceptual understanding, procedural skill and fluency, and application.

Grade-level examples of these three major principles are indicated throughout this document.

Cluster headings can be viewed as the most effective way to communicate the focus and coherence of the standards. The instructional focus must be based on the depth of these ideas, the time needed to master the clusters, and their importance to future mathematics or the later demands of preparing for college and careers.

Teachers and administrators alike should note that the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences presented in a coherent manner. The West Virginia College- and Career-Readiness Standards for Mathematics are learning goals for students that must be mastered by the end of the grade six academic year for students to be prepared for the mathematics content at the grade seven level.

Connecting Mathematical Habits of Mind and Content

The Mathematical Habits of Mind (**MHM**) are developed throughout each grade and, together with the content standards, prescribe that students experience mathematics as a rigorous, coherent, useful, and logical subject. The Mathematical Habits of Mind represent a picture of what it looks like for students to understand and do mathematics in the classroom and should be integrated into every mathematics lesson for all students. The Mathematical Habits of Mind are the behaviors and dispositions of mathematics and should be integrated into every mathematics lesson for all students, and are part of the comprehensive approach to middle and secondary learning per *WVBE Policy, 2510, Assuring Quality of Education: Regulations for Education Programs*.

Although the description of the Mathematical Habits of Mind remains the same at all grade levels, the way these standards develop as students engage with and master new and more advanced mathematical ideas does change. The following table presents examples of how the Mathematical Habits of Mind may be integrated into tasks appropriate for students in grade six.

Mathematical Habits of Mind	Explanation and Examples
MHM1 Make sense of problems and persevere in solving them.	In grade six, students solve real-world problems through the application of algebraic and geometric concepts. These problems involve ratio, rate, area, and statistics. Students seek the meaning of a problem and look for efficient ways to represent and solve it. They may check their thinking by asking themselves questions such as these: “What is the most efficient way to solve the problem?” “Does this make sense?” “Can I solve the problem in a different way?” Students can explain the relationships between equations, verbal descriptions, and tables and graphs. Mathematically proficient students check their answers to problems using a different method.
MHM2 Reason abstractly and quantitatively.	Students represent a wide variety of real-world contexts by using rational numbers and variables in mathematical expressions, equations, and inequalities. Students contextualize to understand the meaning of the number or variable as related to the problem and decontextualize to operate with symbolic representations by applying properties of operations or other meaningful moves. To reinforce students’ reasoning and understanding, teachers might ask, “How do you know?” or “What is the relationship of the quantities?”
MHM3 Construct viable arguments and critique the reasoning of others.	Students construct arguments with verbal or written explanations accompanied by expressions, equations, inequalities, models, graphs, tables, and other data displays (e.g., box plots, dot plots, histograms). They further refine their mathematical communication skills through mathematical discussions in which they critically evaluate their own thinking and the thinking of other students. They pose questions such as these: “How did you get that?” “Why is that true?” “Does that always work?” They explain their thinking to others and respond to others’ thinking.

Mathematical Habits of Mind	Explanation and Examples
MHM4 Model with mathematics.	In grade six, students model problem situations symbolically, graphically, in tables, contextually, and with drawings of quantities as needed. Students form expressions, equations, or inequalities from real-world contexts and connect symbolic and graphical representations. They begin to explore covariance and represent two quantities simultaneously. Students use number lines to compare numbers and represent inequalities. They use measures of center and variability and data displays (e.g., box plots and histograms) to draw inferences about and make comparisons between data sets. Students need many opportunities to make sense of and explain the connections between the different representations. They should be able to use any of these representations, as appropriate, and apply them to a problem context. Students should be encouraged to answer questions such as "What are some ways to represent the quantities?" or "What formula might apply in this situation?"
MHM5 Use appropriate tools strategically.	When solving a mathematical problem, students consider available tools (including estimation and technology) and decide when particular tools might be helpful. For instance, students in grade six may decide to represent figures on the coordinate plane to calculate area. Number lines are used to create dot plots, histograms, and box plots to visually compare the center and variability of the data. Visual fraction models can be used to represent situations involving division of fractions. Additionally, students might use physical objects or applets to construct nets and calculate the surface area of three-dimensional figures. Students should be encouraged to answer questions such as "What approach did you try first?" or "Why was it helpful to use _____?"
MHM6 Attend to precision.	Students continue to refine their mathematical communication skills by using clear and precise language in their discussions with others and in their own reasoning. Students use appropriate terminology when referring to rates, ratios, geometric figures, data displays, and components of expressions, equations, or inequalities. When using ratio reasoning in solving problems, students are careful about specifying units of measure and labeling axes to clarify the correspondence with quantities in a problem. Students also learn to express numerical answers with an appropriate degree of precision when working with rational numbers in a situational problem. Teachers might ask, "What mathematical language, definitions, or properties can you use to explain _____?"
MHM7 Look for and make use of structure.	Students routinely seek patterns or structures to model and solve problems. For instance, students notice patterns that exist in ratio tables, recognizing both the additive and multiplicative properties. Students apply properties to generate equivalent expressions (e.g., $6 + 3x = 3(2 + x)$ by the distributive property) and solve equations (e.g., $2c + 3 = 15$, $2c = 12$ by the subtraction property of equality, $c = 6$ by the division property of equality). Students compose and decompose two- and three-dimensional figures to solve real-world problems involving area and volume. Teachers might ask, "What do you notice when _____?" or "What parts of the problem might you eliminate, simplify, or _____?"

Mathematical Habits of Mind	Explanation and Examples
MHM8 Look for and express regularity in repeated reasoning.	In grade six, students use repeated reasoning to understand algorithms and make generalizations about patterns. During opportunities to solve and model problems designed to support generalizing, they notice that $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ and construct other examples and models that confirm their generalization. Students connect place value and their prior work with operations to understand algorithms to fluently divide multi-digit numbers and perform all operations with multi-digit decimals. Students informally begin to make connections between covariance, rates, and representations that show the relationships between quantities. Students should be encouraged to answer questions such as, "How would we prove that _____?" or "How is this situation like and different from other situations?"

Adapted from ADE 2010, North Carolina Department of Public Instruction (NCDPI) 2013b, and Georgia Department of Education

Standards-Based Learning at Grade Six

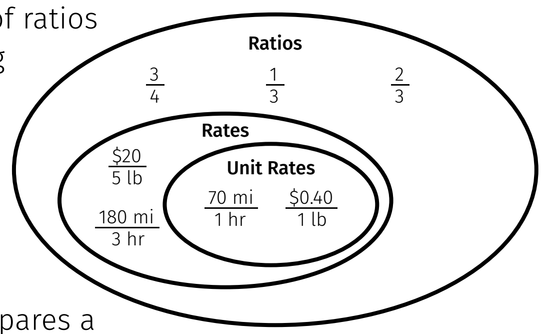
The following narrative is organized by domain from the West Virginia College- and Career-Readiness Standards for Mathematics in grade six and also highlights some necessary foundational skills from previous grade levels. It also provides exemplars to explain the content standards, highlights connections to the Mathematical Habits of Mind (**MHM**), and demonstrates the importance of developing conceptual understanding, procedural skill and fluency, and application.

Domain: Ratios and Proportional Relationships

A critical area of instruction in grade six is to connect ratio, rate, and percentage to whole number multiplication and division and use concepts of ratio and rate to solve problems. Students' prior understanding of and skill with multiplication, division, and fractions contribute to their study of ratios, proportional relationships, unit rates, and percentages. In grades seven and eight, these concepts will extend to include scale drawings, slope, and real-world percent problems.

Ratios and Proportional Relationships
Understand ratio concepts and use ratio reasoning to solve problems. **M.6.1** Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. (e.g., "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes.") **M.6.2** Understand the concept of a unit rate a/b associated with a ratio a:b with b ≠ 0, and use rate language in the context of a ratio relationship. (e.g., "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is 3/4 cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger.") Instructional Note: Expectations for unit rates in this grade are limited to non-complex fractions.

In grade six, a **ratio** is a pair of non-negative numbers, A:B, in a multiplicative relationship. The quantities A and B are related by a rate *r*, where A = *r*B. The number *r* is called a **unit rate** and is computed by $r = \frac{A}{B}$ as long as B ≠ 0. (**M.6.1**) Although the introduction of ratios in grade six involves only non-negative numbers, ratios involving negative numbers are important in algebra and calculus. For example, a student in grade eight understands that if the slope of a line is –2, that means the ratio of rise to run is –2; the y-coordinate decreases by 2 when the x-coordinate increases by 1. In calculus, a negative rate of change means a function is decreasing.



Students work with models to develop their understanding of ratios (**MHM2, MHM6**). Initially, students do not express ratios using fraction notation; this is to allow students to differentiate ratios from fractions and rates. In grade six, students also learn that ratios can be expressed in fraction notation but are different from fractions in several ways. For example, in a litter of 7 puppies, 3 of them are white and 4 of them are black. The ratio of white puppies to black puppies is 3:4. But the fraction of white puppies is not $\frac{3}{4}$; it is $\frac{3}{7}$. A fraction compares a part to the whole, while a ratio can compare either a part to a part or a part to a whole.

Ratios have associated rates. For example, in the ratio 3 cups of orange juice to 2 cups of fizzy water, the rate is $\frac{3}{2}$ cups of orange juice per 1 cup of fizzy water. The term unit rate refers to the numerical part of the rate; in the previous example, the unit rate is the number $\frac{3}{2}$ = 1.5. (The word unit is used to highlight the 1 in "per 1 unit of the second quantity.") Students understand the concept of a unit rate $\frac{a}{b}$ associated with a ratio a:b (with a, b ≠ 0), and use rate language in the context of a ratio relationship. (**M.6.2**)

Examples of Ratio Language	M.6.2

1. If a recipe calls for a ratio of 3 cups of flour to 4 cups of sugar, then the ratio of flour to sugar is 3:4. This can also be expressed with units included, as in "3 cups flour to 4 cups sugar." The associated rate is "$\frac{3}{4}$ cup of flour per cup of sugar." The unit rate is the number $\frac{3}{4}$ = .75.

2. If the soccer team paid \$75 for 15 hamburgers, then this is a ratio of \$75 to 15 hamburgers or 75:15. The associate rate is \$5 per hamburger. The unit rate is the number $\frac{75}{15}$ = 5.

Students understand that rates always have units associated with them that are reflective of the quantities being divided. Common unit rates are cost per item or distance per time. **In grade six, the expectation is that student work with unit rates is limited to fractions in which both the numerator and denominator are whole numbers.** Grade six students use models and reasoning to find rates and unit rates.

Students understand ratios and their associated rates by building on their prior knowledge of division concepts.

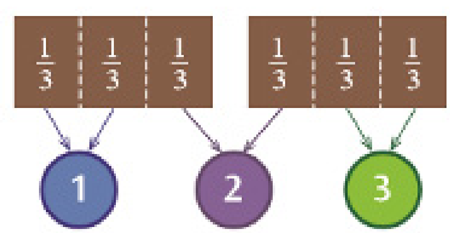
Why must *b* not be equal to 0? **M.6.1**

For a unit rate, or any rational number $\frac{a}{b}$, the denominator b must not equal 0 because division by 0 is *undefined* in mathematics. To see that division by zero cannot be defined in a meaningful way, we relate division to multiplication. That is, if $a \neq 0$ and if $\frac{a}{0} = x$ for any x, there is no x that makes the equation $a = 0 \cdot x$ true. For a different reason, $\frac{0}{0}$ is undefined because it cannot be assigned a unique value. Indeed, if $\frac{0}{0} = x$, then $0 = 0 \cdot x$. which is true for any value of x. So, what would x be?

Examples	M.6.2 (MHM2, MHM6)

There are 2 brownies for 3 students. What is the amount of brownie that each student receives? What is the unit rate?

Solution:
This can be modeled to show that there are $\frac{2}{3}$ of a brownie for each student. The unit rate in this case is $\frac{2}{3}$. In the illustration at the right, each student is counted as he or she receives a portion of brownie, and it is clear that each student receives $\frac{2}{3}$ of a brownie.



Adapted from Kansas Association of Teachers of Mathematics (KATM) 2012, 6th Grade Flipbook

In general, students should be able to identify and describe any ratio using language such as "For every __________, there are __________." For example, for every three students, there are two brownies. (adapted from NCDPI 2013b)

Ratios and Proportional Relationships

Understand ratio concepts and use ratio reasoning to solve problems.

M.6.3

Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

a. Make tables of equivalent ratios relating quantities with whole number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.
b. Solve unit rate problems including those involving unit pricing and constant speed. (e.g., If it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?)
c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.
d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

Students make tables of equivalent ratios relating quantities with whole number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. They use tables to compare ratios.

Grade six students work with tables of quantities in equivalent ratios (also called ratio tables) and practice using ratio and rate language to deepen their understanding of what a ratio describes. As students generate equivalent ratios and record ratios in tables, they should notice the role of multiplication and division in how entries are related to each other. Students also understand that equivalent ratios have the same unit rate. Tables that are arranged vertically may help students to see the multiplicative relationship between equivalent ratios and help them avoid confusing ratios with fractions (adapted from the University of Arizona [UA] Progressions Documents 2011c).

Example: Representing Rations in Different Ways **M.6.3a**

A juice recipe calls for 5 cups of grape juice for every two cups of peach juice. How many cups of grape juice are needed for a batch that uses 8 cups of peach juice?

Use Ratio Reasoning:

"For every 2 cups of peach juice, there are 5 cups of grape juice, so I can draw groups of the mixture to figure out how much grape juice I would need."

(In the illustrations below,  represents 1 cup of grape juice and  represents 1 cup of peach juice.)

"It is easy to see that when you have 4 x 2 = 8 cups of peach juice, you need 4 x 5 = 20 cups of grape juice."

Using a Table:
"I can set up a table. That way it's easy to see that every time I add 2 more cups of peach juice, I need to add 5 cups of grape juice."

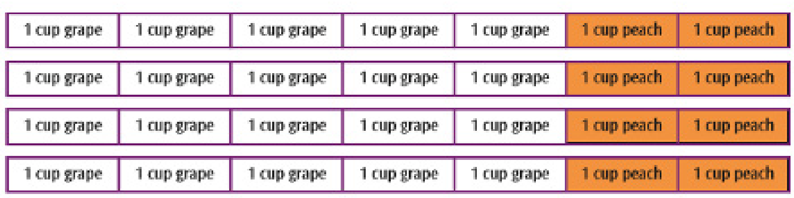
Cups of Grape Juice	Cups of Peach Juice
5	2
10	4
15	6
20	8
15	10

+5 +2

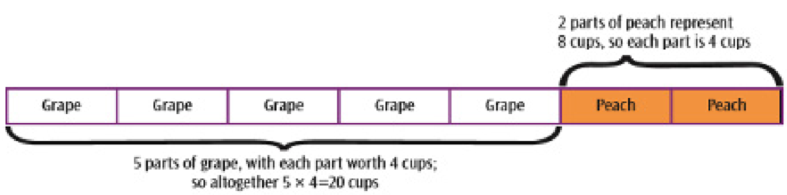
A **tape diagram** (a drawing that looks like a segment of tape) can be used to illustrate a ratio. Tape diagrams are best used when the quantities in a ratio have the same units. A double number line diagram sets up two number lines with zeros connected. The same tick marks are used on each line, but the number lines have different units, which is central to how double number lines exhibit a ratio. **Double number lines** are best used when the quantities in a ratio have different units. The following examples show how tape diagrams and double number lines can be used to solve the problem from the previous example (adapted from UA Progressions Documents 2011c).

Representing Ratios with Tape Diagrams and Double Number Line Diagrams **M.6.3**

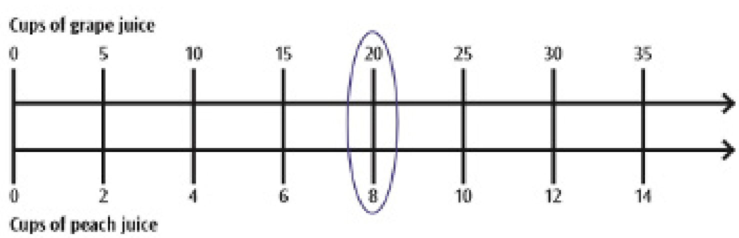
Using a Tape Diagram (Beginning Method):
"I set up a tape diagram. I used pieces of tape to represent 1 cup of liquid. Then I copied the diagram until I had 8 cups of peach juice."



Using a Tape Diagram (Advanced Method):
"I set up a tape diagram in a ratio of 5:2. Since I know there should be 8 cups of peach juice, each section of the tape is worth 4 cups. That means there are 5 x 4 = 20 cups of grape juice."



Using a Double Number Line Diagram:
"I set up a double number line, with cups of grape juice on the top and cups of peach juice on the bottom. When I counted up to 8 cups of peach juice, I see that this brings me to 20 cups of grape juice."



Representing ratios in several ways can help students see the additive and multiplicative structure of ratios (**MHM7**). Standard **M.6.3a** calls for students to create tables of equivalent ratios and represent the resulting data on a coordinate grid. Eventually, students see this additive and multiplicative structure in the graphs of ratios, which will be useful in grade eight when studying slopes and linear functions. (Refer to standard **M.6.20** as well.)

Representing Ratios using Tables and Graphs **M.6.3a**

Additive Structure

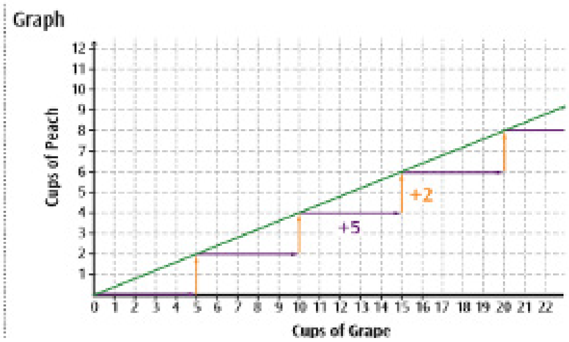
Table

Cups of Grape	Cups of Peach
5	2
10	4
15	6
20	8
25	10

+5 +5 +5 +5

+2 +2 +2 +2

Graph



Multiplicative Structure

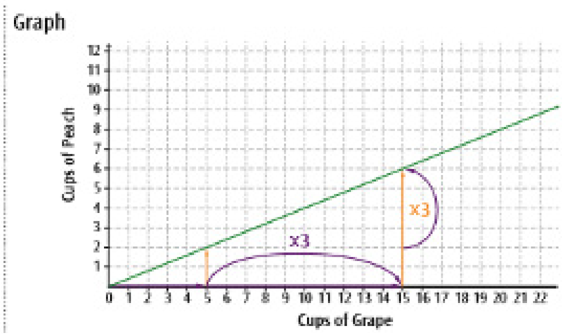
Table

Cups of Grape	Cups of Peach
5	2
10	4
15	6
20	8
100	40

×2 ×3 ×20

×2 ×3 ×20

Graph



Adapted from UA Progressions Documents 2011c.

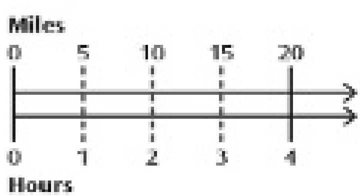
As students solve similar problems, they develop their skills in several of the Mathematical Habits of Mind, reasoning abstractly and quantitatively (**MHM2**), abstracting information from the problem, creating a mathematical representation of the problem, and correctly working with both part–part and part–whole situations. Students model with mathematics (**MHM4**) as they solve these problems by using tables and/or ratios. They attend to precision (**MHM6**) as they properly use ratio notation, symbolism, and label quantities. The following table presents a sample classroom activity that connects the West Virginia College- and Career-Readiness Standards and the Mathematical Habits of Mind. The activity is appropriate for students who have already been introduced to ratios and associated rates.

Standard **M.6.3b–d** calls for students to apply their newfound ratio reasoning to various problems in which ratios appear, including problems involving unit price, constant speed, percent, and the conversion of measurement units. In grade six, generally only whole number ratios are considered. The basic idea of percent is a particularly relevant and important topic for young students to learn, as they will use this concept throughout their lives (**MHM4**). Percent is discussed in a separate section that follows. Below are several more examples of ratios and the reasoning expected in the domain, Ratios and Proportional Relationships.

Examples of Problems Involving Ratio Reasoning — M.6.3b, M.6.3d

1. On a bicycle, you can travel 20 miles in 4 hours. At the same rate, what distance can you travel in 1 hour?

Solution:
Students might use a double number line diagram to represent the relationship between miles ridden and hours elapsed. They build on fraction reasoning from earlier grades to divide the double number line into 4 equal parts and mark the double number line accordingly. It becomes clear that in 1 hour, a person can ride 5 miles, which is a rate of 5 miles per hour.

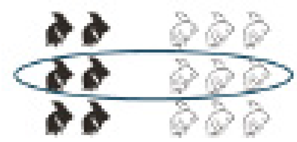


(M.6.3d)

2. At the pet store, a fish tank has guppies and goldfish in a ratio of 6:9. Show that this is the same as the ratio of 2:3.

Solution:
Students should be able to find equivalent ratios by drawing pictures or using ratio tables. A ratio of 6:9 might be represented in the following way, with black fish as guppies and white fish as goldfish:

This picture can be rearranged to show 3 sets of 2 guppies and 3 sets of 3 goldfish, for a ratio of 2:3.



(M.6.3b)

3. Use the information in the following table to find the number of yards that equals 24 feet.

Feet	3	6	9	15	24
Yards	1	2	3	5	?

Solution:
Students can solve this in several ways.

1. They can observe the associated rate from the table, 3 feet per yard, and they can use multiplication to see that 24 feet = 8 × 3 feet, so the answer 8 × 1 yard, or 8 yards.
2. They can notice that 24 feet = 4 × (6 feet), so the answer is 4 × (2 yards) = 8 yards.
3. They can see that ratios, you can add entries in a table because of the distributive property:
   9 feet + 15 feet = 24 feet
   3 (3 feet) + 5 (3 feet) = 8 (3 feet)

And since 3 feet = 1 yard, the correct answer is 8 yards.

(M.6.3d)

4. The cost of 3 cans of pineapple at Superway Store is $2.25, and the cost of 6 cans of the same kind of pineapple is $4.80 at Grocery Giant. Which store has the better price for pineapple?

Solution:
Students can solve this in several ways.

1. They can make a table that lists prices for different numbers of cans and compare the price for the same number of cans.
2. They can multiply the number of cans and their price at Superway Store by 2 to see that 6 cans there cost $4.50, so the same number of cans cost less at Superway Store than at Grocery Giant (where 6 cans cost $4.80).
3. Finally, they can find the unit price at each store:

   $\frac{(\$2.25)}{3}$ =$.75 per can at Superway Store

   $\frac{(\$4.80)}{6}$ =$.80 per can at Grocery Giant

(M.6.3b)

Percent: A Special Type of Rate

Standard **M.6.3c** calls for grade six students to understand percent as a special type of rate, and students use models and tables to solve percent problems. This is students' first formal introduction to percent. Students understand that percentages represent a rate per 100; for example, to find 75% of a quantity means to multiply the quantity by $\frac{3}{4}$ or, equivalently, by the fraction $\frac{75}{100}$. The use of tables, tape diagrams, and double line diagrams in representing percent problems can help students understand this concept. Understanding of percent is related to students' understanding of fractions and decimals. A thorough understanding of place value helps students see the connection between decimals and percent (for example, students understand that 0.30 represent $\frac{30}{100}$, which is the same as 30%).

Students can use simple "benchmark percentages" (e.g., 1%, 10%, 25%, 50%, 75%, or 100%) as one strategy for solving percent problems. By using the distributive property to reason about rates, students see that percentages can be combined to find other percentages, and thus benchmark percentages become a very useful tool when learning about percent (**MHM5**).

Benchmark Percentages	**M.6.3c (MHM7)**

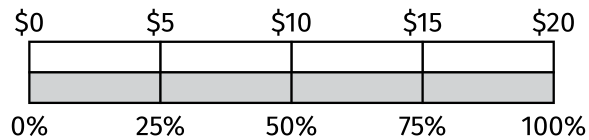
- 100% of a quantity is the entire quantity, or "1 times" the quantity.
- 50% of a quantity is half the quantity (since 50% = $\frac{50}{100} = \frac{1}{2}$), and 25% is one-quarter of a quantity (since 25% = $\frac{25}{100} = \frac{1}{4}$).
- 10% of a quantity is $\frac{1}{10}$ of the quantity (since 10% = $\frac{10}{100} = \frac{1}{10}$), so to find 10% of a quantity, students can divide the quantity by 10. Similarly, 1% is 1/100 of a quantity.
- 200% of a quantity is twice the quantity (since 200% = $\frac{200}{100}$ = 2).
- 75% of a quantity is $\frac{3}{4}$ of the quantity. Students also find that 75% = 50% + 25%, or 75% = 3 x 25%. Tape diagrams and double number lines can be useful for seeing this relationship.

A ***percent bar*** is a visual model, similar to a combined double number line and tape diagram, which can be used to solve percent problems. Students can fold the bar to represent benchmark percentages such as 50% (half), 25% and 75% (quarters), and 10% (tenths). Teachers should connect percent to ratios so that students see percent as a useful application of ratios and rates.

Examples: Connecting Percent to Ratio Reasoning **M.6.3c**

Andrew was given an allowance of $20. He used 75% of his allowance to go to the movies. How much money was spent at the movies?

Solution: "By setting up a percent bar, I can divide the $20 into four equal parts. I see that he spent $15 at the movies:



What percent is 12 of out 25?

Solutions: [a] "I set up a simple table and found that 12 out of 25 is the same as 24 out of 50, which is the same as 48 out of 100. So 12 out of 25 is 48%."

Part	12	24	48
Whole	25	50	100

[b] "I saw that 4 × 25 is 100, so I found 4 × 12 = 48. So 12 out of 25 is the same as 48 out of 100, or 48%."
[c] "I know that I can divide 12 by 25, since $\frac{12}{25}$ = 12 ÷ 25. I got 0.48, which is the same as $\frac{48}{100}$, or 48%."

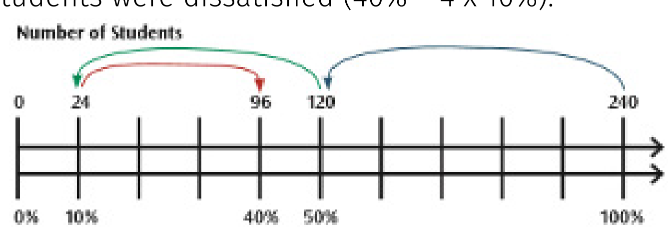
Adapted from ADE 2010 and NCDPI 2013b.

There are several types of percent problems that students should become familiar with, including finding the percentage represented by a part of a whole, finding the unknown part when given a percentage and whole, and finding an unknown whole when a percentage and part are given. The following examples illustrate these problem types, as well as how to use tables, tape diagrams, and double number lines to solve them. **Students in grade six are not responsible for solving multi-step percent problems such as finding sales tax, markups and discounts, or percent change.**

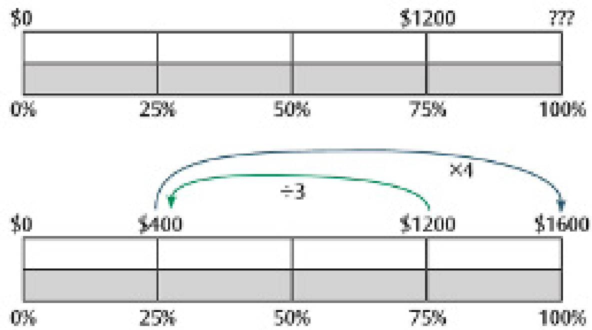
More Examples of Percent Problems **M.6.3c**

Finding an Unknown Part. Last year, Mr. Christian's class had 30 students. This year, the number of students in his class is 150% of the number of students he had in his class last year. How many students does he have this year?
Solution:
"Since 100% is 30 students, I know that 50% is 30 ÷ 2 = 15 students. This means that 150% is 3 x 15 = 45 students, since 150% is 3 x 50%. His class is made up of 45 students this year."

Finding an Unknown Percentage. When all 240 grade six students were polled, results showed that 96 students were dissatisfied with the music played at a school dance. What percentage of sixth-grade students does this represent?
Solution:
"I set up a double number line diagram. It was easy to find that 50% was 120 students. This meant that 10% was 120 ÷ 5 = 24 students. I noticed that 96 ÷ 24 is 4. Reading my double number line, this means that 40% of the students were dissatisfied (40% = 4 x 10%)."



Finding an Unknown Whole. If 75% of the budget is $1,200, what is the full budget?
Solution:
"By setting up a fraction bar, I can find 25%, since I know 75% is $1,200. Then, I multiply by 4 to give me 100%. Since 25% is $400, I see that 100% is $1,600."



In problems such as this one, teachers can use scaffolding questions such as these:
• If you know 75% of the budget, how can we determine 25% of the budget?
• If you know 25% of the budget, how can this help you find 100% of the budget?

Source: UA Progressions Documents 2011c

When students have had sufficient practice solving percent problems with tables and diagrams, they can be led to represent percentages as decimals to solve problems. For instance, the previous three problems can be solved using methods such as those shown below.

Examples **M.6.3c**

If the class has 30 students, then 150% can be found by finding the fraction:

$$\frac{150}{100} = \frac{15}{10} = 1.5$$

$$1.5 \times 30 = 45$$

So the answer is 45 students.

Since 96 out of 240 students were dissatisfied with the music at the dance, this means that:

$$\frac{96}{240} = 0.4$$

$$0.4 = 40\%$$

40% were dissatisfied with the music.

Since the budget is unknown, let's call it **B**. Then we know that 75% of the budget is $1,200, which means that 0.75**B** = 1200. This can be solved by finding **B** = 1200 ÷ 0.75
Alternately, students may see that:

$\frac{75}{100} = \frac{1200}{b}$, which can be rewritten as $\frac{3}{4} = \frac{1200}{b}$

By reasoning with equivalent fractions, since $\frac{3}{4} = \frac{(3 \cdot 400)}{(4 \cdot 400)} = \frac{1200}{1600}$, we see that **B** = $1600.

Percent problems give students opportunities to develop Mathematical Habits of Mind as they use a variety of strategies to solve problems, use tables and diagrams to represent problems (**MHM4**), and reason about percent (**MHM1, MHM2**).

Common Misconceptions: Ratios and Fractions

- Although ratios can be represented as fractions, the connection between ratios and fractions is subtle. Fractions express a part-to-whole comparison, but ratios can express part-to-whole or part-to-part comparisons. Care should be taken if teachers choose to represent ratios as fractions at this grade level.
- Proportional situations can have several ratios associated with them. For instance, in a mixture involving 1 part juice to 2 parts water, there is a ratio of 1 part juice to 3 total parts (1:3), as well as the more obvious ratio of 1:2.
- Students must carefully reason about why they can add ratios. For instance, in a mixture with lemon drink and fizzy water in a ratio of 2:3, mixtures made with ratios 2:3 and 4:6 can be added to give a mixture of ratio 6:9, equivalent to 2:3. This is because the following are true:

  2 (parts lemon drink) + 4 (parts lemon drink) = 6 (parts lemon drink)
  3 (parts fizzy water) + 6 (parts fizzy water) = 9 (parts fizzy water)

  However, one would never add fractions by adding numerators and denominators:
  $\frac{2}{3} + \frac{4}{6} \neq \frac{6}{9}$

Domain: The Number System

In grade six, students strengthen their understanding of division of fractions and extend the concept of *number* to the system of rational numbers, which includes negative numbers. By the end of sixth grade, students complete the understanding of fractions. Students also work toward fluency with multi-digit division and multi-digit decimal operations.

The Number System
Apply and extend previous understandings of multiplication and division to divide fractions by fractions.
M.6.4 Interpret and compute quotients of fractions and solve word problems involving division of fractions by fractions by using visual fraction models and equations to represent the problem.(e.g., Create a story context for (2/3) ÷ (3/4) and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that (2/3) ÷ (3/4) = 8/9 because 3/4 of 8/9 is 2/3. (In general, (a/b) ÷ (c/d) = ad/bc.) How much chocolate will each person get if 3 people share 1/2 lb. of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?)

In grade five, students learn to divide whole numbers by unit fractions and unit fractions by whole numbers. These experiences lay the conceptual foundation for understanding general methods of division of fractions in sixth grade. Grade six students continue to develop division by using visual models and equations to divide fractions by fractions to solve word problems (**M.6.4**). Student understanding of the meaning of operations with fractions builds upon the familiar understandings of these meanings with whole numbers and can be supported with visual representations. To help students make this connection, teachers might have students think about a simpler problem with whole numbers and then use the same operation to solve with fractions.

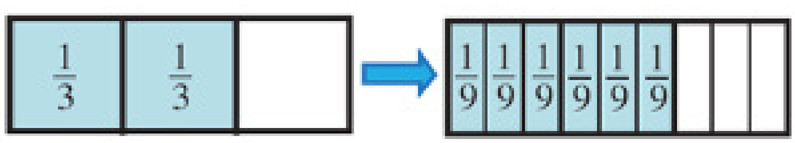
Looking at a problem through the lens of "How many groups?" or "How many in each group?" helps students visualize what is being sought. Encourage students to explain their thinking and to recognize division in two different situations: measurement division, which requires finding how many groups (e.g., how many groups can you make?); and fair-share division, which requires equal sharing (e.g., finding how many are in each group). In grade five, students represented division problems like $4 \div \frac{1}{2}$ with diagrams and reasoned why the answer is 8 (e.g., how many halves are in 4?). They may have discovered that $4 \div \frac{1}{2}$ can be found by multiplying 4×2 (i.e., each whole gives 2 halves, so there are 8 halves altogether). Similarly, students may have found that $\frac{1}{3} \div 5 = \frac{1}{3} \times \frac{1}{5}$. These generalizations will be used when students develop general methods for dividing fractions. Teachers should be aware that making visual models for general division of fractions can be difficult. Teachers need to be mindful when selecting fractions for early work. They should be easy for the students to draw. Once students have an understanding of the concepts, the teacher may discuss general methods for dividing fractions and use these methods to solve problems.

The following examples illustrate how reasoning about division can help students understand fraction division before they move on to general methods.

Representing Ratios using Tables and Graphs	**M.6.4**

1. Three people share $\frac{2}{3}$ of a pound of watermelon. How much watermelon does each person get?

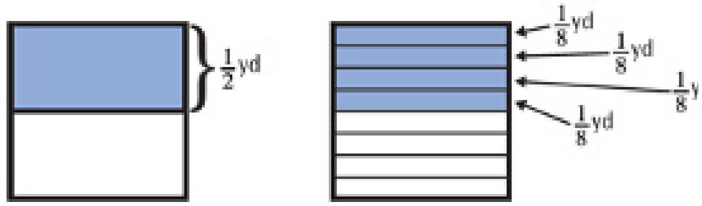
Solution:
This problem can be represented by $\frac{2}{3} \div 3$. To solve it, students might represent the watermelon with a diagram such as the one below. There are two $\frac{1}{3}$ - pound pieces represented in the picture. Students can see that $\frac{1}{3}$ divided among three people is $\frac{1}{9}$. Since there are 2 such pieces, each person receives $\frac{2}{9}$ of a pound of watermelon.



Problems like this one can be used to support the fact that, in general, $\frac{a}{b} \div c = \frac{a}{b} \times \frac{1}{c}$.

2. Manny has $\frac{1}{2}$ of a yard of fabric with which he intends to make bookmarks. Each bookmark is made from $\frac{1}{8}$ of a yard of fabric. How many bookmarks can Manny make?

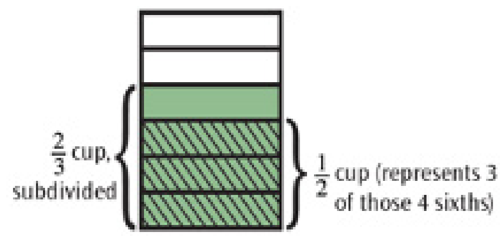
Solution:
Students can think, "How many $\frac{1}{8}$-yard pieces can I make from $\frac{1}{2}$ of a yard of fabric?" By subdividing the $\frac{1}{2}$ of a yard of fabric into eighths (of a yard), students can see that there are 4 such pieces.



This problem can be represented by $\frac{1}{2} \div \frac{1}{8}$. Problems like this one can be used to support the fact that, in general, $\frac{1}{b} \div \frac{1}{d} = \frac{1}{b} \times d$.

3. You are making a recipe that calls for $\frac{2}{3}$ of a cup of yogurt. You have $\frac{1}{2}$ cup of yogurt from a snack pack. How much of the recipe can you make?

Solutions:
Students can think, "How many $\frac{2}{3}$ cup portions can be made from $\frac{1}{2}$ cup?"

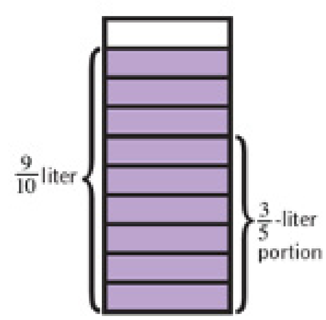


Students can reason that the answer will be less than 1, as there is not enough yogurt to make 1 full recipe. The difficulty with this problem is that it is not immediately apparent how to find thirds from halves. Students can convert the fractions into ones with common denominators to make the problem more accessible. Since $\frac{2}{3} = \frac{4}{6}$ and $\frac{1}{2} = \frac{3}{6}$ it makes sense to represent the $\frac{2}{3}$ cup required for the recipe divided into $\frac{1}{6}$ cup portions. As the diagram shows, the recipe calls for $\frac{4}{6}$ cup, but there are only 3 of the 4 sixths that are available. Each sixth is $\frac{1}{4}$ of a recipe, and we have 3 of them, so we can make $\frac{3}{4}$ of a recipe. This problem can be represented by $\frac{1}{2} \div \frac{2}{3}$.

Problems like this one can be used to support the division-by-common-denominators strategy.

4. A certain type of water bottle holds $\frac{3}{5}$ of a liter of liquid. How many of these bottles could be filled from $\frac{9}{10}$ of a liter of juice?

Solution:
The picture shows $\frac{9}{10}$ of a liter of juice. Since 6 tenths make $\frac{3}{5}$ of a liter, it is clear that one bottle can be filled. The remaining $\frac{3}{10}$ of a liter represents $\frac{1}{2}$ of a bottle, so it makes sense to say that $1\frac{1}{2}$ bottles could be filled. Notice that $\frac{1}{10} \div \frac{1}{5} = \frac{1}{2}$, meaning that there is one-half of $\frac{1}{5}$ in each $\frac{1}{10}$. This means that in 9 tenths, there are 9 halves of $\frac{1}{5}$. But since the capacity of a bottle is 3 of these fifths, there are $\frac{9}{2} \div 3 = \frac{3}{2}$ of these bottles. This line of reasoning supports the idea that numerators and denominators can be divided — that is, $\frac{9}{10} \div \frac{3}{5} = \frac{(9 \div 3)}{(10 \div 5)} = \frac{3}{2}$.



Adapted from ADE 2010, NCDPI 2013b, and KATM 2012, 6th Grade Flipbook.

Common Misconceptions

Students may confuse dividing a quantity by $\frac{1}{2}$ with dividing a quantity in half. Dividing by $\frac{1}{2}$ is finding how many $\frac{1}{2}$-sized portions there are, as in "dividing 7 by $\frac{1}{2}$," which is $7 \div \frac{1}{2} = 14$. On the other hand, to divide a quantity in half is to divide the quantity into two parts equally, as in "dividing 7 in half" yields $\frac{7}{2} = 3.5$. Students should understand that dividing in half is the same as dividing by 2.

Adapted from KATM 2012, 6th Grade Flipbook.

Students should also connect division of fractions with multiplication. For example, in the problems above, students should reason that $\frac{2}{3} \div 3 = \frac{2}{9}$, since $3 \times \frac{2}{9} = \frac{2}{3}$. Also, it makes sense that $\frac{1}{2} \div \frac{1}{8} = 4$, since $\frac{1}{8} \times 4 = \frac{1}{2}$, and that $\frac{1}{2} \div \frac{2}{3} = \frac{3}{4}$ because $\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$. The relationship between division and multiplication is used to develop general methods for dividing fractions.

General Methods for Dividing Fractions

1. **Finding common denominators.** Interpreting division as measurement division allows one to divide fractions by finding common denominators (i.e., *common denominations*). For example, to divide $\frac{7}{8} \div \frac{2}{5}$, students need to find a common denominator, so $\frac{7}{8}$ is rewritten as $\frac{35}{40}$ and $\frac{2}{5}$ is rewritten as $\frac{16}{40}$. Now the problem becomes, "How many groups of 16 fortieths can we get out of 35 fortieths?" That is, the problem becomes $35 \div 16 = \frac{35}{16}$. This approach of finding common denominators reinforces the linguistic connection between *denominator* and *denomination*.

2. **Dividing numerators and denominators (special case).** By thinking about the relationship between division and multiplication, students can reason that a problem like $\frac{8}{15} \div \frac{2}{5} = ?$ is the same as finding $\frac{2}{5} \times ? = \frac{8}{15}$. Students can see that the fraction $\frac{4}{3}$ represents the missing factor, but this is the same result as if one simply divided numerators and denominators: $\frac{8}{15} \div \frac{2}{5} = \frac{(8 \div 2)}{(15 \div 5)} = \frac{4}{3}$. Although this strategy works in general, it is particularly useful when the numerator and denominator of the divisor are factors of the numerator and denominator of the dividend, respectively.

3. **Dividing numerators and denominators (leading to the general case).** By rewriting fractions as equivalent fractions, students can use the previous strategy in other cases — for instance, when the denominator of the divisor is not a factor of the denominator of the dividend. For example, when finding $\frac{2}{3} \div \frac{2}{7}$, students can think of $\frac{2}{3}$ as $\frac{14}{21} = \frac{(2 \times 7)}{(3 \times 7)}$, to arrive at:

$$\begin{aligned} &\frac{2}{3} \div \frac{2}{7} \\ &= \frac{14}{21} \div \frac{2}{7} \\ &= \frac{(14 \div 2)}{(21 \div 7)} \\ &= \frac{7}{3} \end{aligned}$$

4. **Dividing numerators and denominators (general case).** When neither the numerator nor the denominator of the divisor is a factor of those of the dividend, equivalent fractions can be used again to develop a strategy. For instance, with a problem like $\frac{3}{4} \div \frac{5}{7}$, the fraction $\frac{3}{4}$ can be rewritten as $\frac{(3 \times 5 \times 7)}{(4 \times 5 \times 7)}$ and then the division can be performed. When the fraction is left in this form, students can see that the following is true:

$$\frac{3}{4} \div \frac{5}{7}$$
$$= \frac{(3 \times 5 \times 7)}{(4 \times 5 \times 7)} \div \frac{5}{7}$$
$$= \frac{((3 \times 5 \times 7) \div 5)}{((4 \times 5 \times 7) \div 7)}$$
$$= \frac{(3 \times 7)}{(4 \times 5)}$$
$$= \frac{3}{4} \times \frac{7}{5}$$

This line of reasoning shows why it makes sense to find the reciprocal of the divisor and multiply to find the result.

Teaching the "multiply-by-the-reciprocal" method for dividing fractions without having students develop an understanding of why it works may confuse students and interfere with their ability to apply division of fractions to solve word problems. Teachers can gradually develop strategies (such as those described above) to help students see that, in general, fractions can be divided in two ways:

» Divide the first fraction (dividend) by the top and bottom numbers (numerator and denominator) of the second fraction (divisor).
» Find the reciprocal of the second fraction (divisor) and then multiply the first fraction (dividend) by it.

The following is an algebraic argument that $\frac{a}{b} \div \frac{c}{d} = x$ precisely when $x = \frac{a}{b} \times \frac{d}{c}$. Starting with $\frac{a}{b} = x \cdot \frac{c}{d}$, it can be argued that if both sides of the equation are multiplied by the multiplicative inverse of $\frac{c}{d}$, x can be isolated on the right. Thus, students examine $\frac{a}{b} \cdot \frac{d}{c} = \left(x \cdot \frac{c}{d}\right) \cdot \frac{d}{c}$. Continuing the computation on the right, students can see that $\left(x \cdot \frac{c}{d}\right) \cdot \frac{d}{c} = x \cdot \left(\frac{c}{d} \cdot \frac{d}{c}\right) = x \cdot 1 = x$. Since $\frac{a}{b} \div \frac{c}{d} = x$ as well, we have $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

The Number System
Compute fluently with multi-digit numbers and find common factors and multiples. **M.6.5** Fluently divide multi-digit numbers using the standard algorithm. **M.6.6** Fluently add, subtract, multiply and divide multi-digit decimals using the standard algorithm for each operation. **M.6.7** Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor (e.g., express 36 + 8 as 4 (9 + 2)).

In previous grades, students build a conceptual understanding of operations with whole numbers and become fluent in multi-digit addition, subtraction, and multiplication. In grade six, students work toward fluency with multi-digit division and multi-digit decimal operations (**M.6.5-M.6.6**). Fluency with the standard algorithms is expected, but an algorithm is defined by its steps, not by the way those steps are recorded in writing, so minor variations in written methods are acceptable.

FLUENCY

The West Virginia College- and Career-Readiness Standards set expectations for fluency in computation (e.g., "Fluently divide multi-digit numbers" (**M.6.5**) and "Fluently add, subtract, multiply, and divide multi-digit decimals" (**M.6.6**) using the standard algorithm). Such standards are culminations of progressions of learning, often spanning several grades, involving conceptual understanding, thoughtful practice, and extra support where necessary. The word fluent is used in the standards to mean "reasonably fast and accurate" and possessing the ability to use certain facts and procedures with enough facility that using such knowledge does not slow down or derail the problem solver as he or she works on more complex problems. Procedural fluency requires skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. Developing fluency in each grade may involve a mixture of knowing some answers, knowing some answers from patterns, and knowing some answers through the use of strategies.

Adapted from UA Progressions Documents 2011a.

Instructional Focus

In grade three, division is introduced conceptually as the inverse of multiplication. In grade four, students continue using place-value strategies, properties of operations, the relationship between multiplication and division, area models, and rectangular arrays to solve problems with one-digit divisors and develop and explain written methods. This work is extended in grade five to include two-digit divisors and all operations with decimals to hundredths. In grade six, fluency with the algorithms for division is reached (**M.6.5**).

Grade six students fluently divide using the standard algorithm (**M.6.5**). Students should examine several methods for recording division of multi-digit numbers and focus on a variation of the standard algorithm that is efficient and makes sense to them. They can compare variations to understand how the same step can be written differently but still have the same place-value meaning. All such discussions should include place-value terms. Students should see examples of standard algorithm division that can be easily connected to place-value meanings.

Example: Scaffold Division — M.6.5

Scaffold division is a variation of the standard algorithm in which partial quotients are written to the right of the division steps rather than above them.

To find the quotient 3440 ÷ 16, students can begin by asking, "How many groups of 16 are in 3440?" This is a measurement interpretation of division and can form the basis of the standard algorithm. Students estimate that there are at least 200 groups of 16, since 2 × 16 = 32, and therefore 200 × 16 = 3200. They would then ask, "How many groups of 16 are in the remaining 240?" Clearly, there are at least 10. The next remainder is then 80 = 240 − 160, and we see that there are 5 more groups of 16 in this remaining 80. The quotient in this strategy is then found to be 200 + 10 + 5 = 215.

Divisor	Dividend	Quotient
16	3440 −3200	200
	240 − 160	10
	80 − 80	5
	0	**215**
	Remainder	**Quotient**

As shown in the next example, the partial quotients may also be written above each other over the dividend. Students may also consider writing single digits instead of totals, provided they can explain why they do so with place-value reasoning, dropping all of the zeros in the quotients and subtractions in the dividend; in the example that follows, students would write “215” step by step above the dividend. In both cases, students use place-value reasoning.

Example: Division Using Single Digits Instead of Totals **M.6.5**

To ensure that students understand and apply place-value reasoning when writing single digits, teachers can ask, “How many groups of 16 are in 34 hundreds?” Since there are two groups of 16 in 34, there are 2 hundred groups of 16 in 34 hundreds, so we record this with a 2 in the hundreds place above the dividend. The product of 2 and 16 is recorded, and we subtract 32 from 34, understanding that we are subtracting 32 hundreds from 34 hundreds, yielding 2 remaining hundreds. Next, when we “bring the 4 down to write 24,” we understand this as moving to the digit in the dividend necessary to obtain a number larger than the divisor. Again, we focus on the fact that there are 24 (tens) remaining, and so the question becomes, “How many groups of 16 are in 24 tens?” The algorithm continues, and the quotient is found.

$$
\begin{array}{rrrrr}
 & & 2 & 1 & 5\\ \hline
16\,) & 3 & 4 & 4 & 0\\
- & 3 & 2 & & \\ \hline
 & & 2 & 4 & \\
 & - & 1 & 6 & \\ \hline
 & & & 8 & 0\\
 & & - & 8 & 0\\ \hline
 & & & & 0
\end{array}
$$

Students should have experience with many examples similar to the two discussed above. Teachers should be prepared to support discussions involving place value if misunderstanding arises. There may be other effective ways for teachers to include place-value concepts when explaining a variation of the standard algorithm for division. Teachers are encouraged to find a method that works for them and their students. The standards support coherence of learning and conceptual understanding, and it is crucial for instruction to build on students’ previous mathematical experiences. Refer to the following example and to the Educators’ Guide for Mathematics: Grade 5 for further explanation of division strategies.

Connecting Division Algorithms and Place Value **M.6.5**

Algorithm 1	Explanation	Algorithm 2	Explanation
$\begin{array}{r} 200 \\ 32\overline{)8456} \end{array}$	There are 200 groups of 32 in 8456.	$\begin{array}{r} 2 \\ 32\overline{)8456} \end{array}$	There are 2 (hundred) groups of 32 in 84 (hundred).
$\begin{array}{r} 200 \\ 32\overline{)8456} \\ -6400 \\ \hline 2056 \end{array}$	200 times 32 is 6400, so we subtract and find there is 2056 left to divide.	$\begin{array}{r} 2 \\ 32\overline{)8456} \\ -64 \\ \hline 205 \end{array}$	2 times 32 is 64, so there are 64 (hundreds) to subtract from 84 (hundreds). We include the 5 with what is left over, since the dividend (205) must be larger than the divisor.

60 200 32 ⟌ 8456 - 6400 2056	There are 60 groups of 32 in 2056.	26 32 ⟌ 8456 - 64 205	Now we see that there are 6 (tens) groups of 32 in 205 (tens).
60 200 32 ⟌ 8456 - 6400 2056 - 1920 136	60 times 32 is 1920, so we subtract and find there is 136 left to divide.	26 32 ⟌ 8456 - 64 205 - 192 136	6 times 32 is 192, so there are 192 (tens) to subtract from 205 (tens). Again, we include the 6 with what is left over since the dividend must be greater than the divisor.
4 60 200 32 ⟌ 8456 - 6400 2056 - 1920 136	There are 4 groups of 32 in 136.	264 32 ⟌ 8456 - 64 205 - 192 136	Now we see that there are 4 groups of 32 in 136.
4 60 200 32 ⟌ 8456 - 6400 2056 - 1920 136 - 128 8	4 times 32 is 128, so we subtract and find 8 left to divide. But since 8 is smaller than the divisor, this is the remainder. So the quotient is 200 + 60 + 4 = 264, with a remainder of 8, or $264\frac{8}{32} = 264\frac{1}{4}$. Another way to say this is 8456=32(264)+8	264 32 ⟌ 8456 - 64 205 - 192 136 - 128 8	4 times 32 is 128, so we subtract and find 8 left to divide. But since 8 is smaller than the divisor, this is the remainder. So the quotient is 264 with a remainder of 8, or $264\frac{8}{32} = 264\frac{1}{4}$. Another way to say this is 8456=32(264)+8

Standard **M.6.6** requires grade six students to fluently apply standard algorithms when working with operations with decimals. In grades four and five, students learn to add, subtract, multiply, and divide decimals (to hundredths) with concrete models, drawings, and strategies and use place value to explain written methods for these operations. In grade six, students become fluent in the use of some written variation of the standard algorithms of each of these operations.

The notation for decimals depends upon the regularity of the place-value system across all places to the left and right of the ones place. This understanding explains why addition and subtraction of decimals can be accomplished with the same algorithms as for whole numbers; like values or units (such as tens or thousandths) are combined. To make sure students add or subtract like places, teachers should provide students with opportunities to solve problems that include zeros in various places and problems in which they might add zeros at the end of a decimal number. For adding and subtracting decimals, a conceptual approach that supports consistent student understanding of place-value ideas might instruct students to line up place values rather than "lining up the decimal point."

Instructional Focus
Students should discuss how addition and subtraction of all quantities have the same basis: adding or subtracting like place-value units (whole numbers and decimal numbers), adding or subtracting like unit fractions, or adding or subtracting like measures. Thus, addition and subtraction are consistent concepts across grade levels and number systems.

In grade five, students multiply decimals to hundredths. They understand that multiplying decimals by a power of 10 "moves" the decimal point as many places to the right as there are zeros in the multiplying power of 10 (see the discussion of standards **M.5.4-M.5.5** in the Grade 5 document). In grade six, students extend and apply their place-value understanding to fluently multiply multi-digit decimals (**M.6.6**). Writing decimals as fractions whose denominator is a power of 10 can be used to explain the "decimal point rule" in multiplication. For example:

$$
\begin{aligned}
&2.4 \times 0.37 \\
&= \frac{24}{10} \times \frac{37}{100} \\
&= \frac{(24 \times 37)}{(10 \times 100)} \\
&= \frac{888}{1000} \\
&= 0.888
\end{aligned}
$$

This logical reasoning based on place value and decimal fractions justifies the typical rule, "Count the decimal places in the numbers and insert the decimal point to make that many places in the product."

The general methods used for computing quotients of whole numbers extend to decimals with the additional concern of where to place the decimal point in the quotient. Students divide decimals to hundredths in grade five, but in grade six they move to using standard algorithms for doing so. In simpler cases, such as 16.8 ÷ 8, students can simply apply the typical division algorithm, paying particular attention to place value. When problems get more difficult (e.g., when the divisor also has a decimal point), then students may need to use strategies involving rewriting the problem through changing place values. Reasoning similar to that for multiplication can be used to explain the rule that "When the decimal point in the divisor is moved to make a whole number, the decimal point in the dividend should be moved the same number of places." For example, for a problem like 4.2 ÷ 0.35, a student might give a rote recipe: "Move the decimal point two places to the right in 0.35 and also in 4.2." Teachers can instead appeal to the idea that a simpler but equivalent division problem can be formed by multiplying both numbers by 100 and still yield the same quotient. That is:

$$4.2 \div 0.35$$
$$= (4.2 \times 100) \div (0.35 \times 100)$$
$$= 420 \div 35$$
$$= 12$$

It is vitally important for teachers to pay attention to students' understanding of place value. There is no conceptual understanding gained by referring to this only as "moving the decimal point." Teachers can refer to this more meaningfully as "multiplying by $\frac{n}{n}$ in the form of $\frac{100}{100}$."

Examples of Decmial Operations **M.6.6**

1. Maria had 3 kilograms of sand for a science experiment. She had to measure out exactly 1.625 kilograms for a sample. How much sand will be left after she measures out the sample?

   Solution:
   Student thinks, "I know that 1.625 is a little more than 1.5, so I should have about 1.5 kilograms remaining. I need to subtract like place values from each other, and I notice that 1.625 has three place values to the right of the ones place, so if I make zeros in the tenths, hundredths, and thousandths places of 3 to make 3.000, then the numbers have the same number of place values. Then it's easier to subtract: 3.000 – 1.625 = 1.375. So, there are 1.375 kilograms left."

2. How many ribbons 1.5 meters long can Victor cut from a cloth that is 15.75 meters long?

   Solution:
   Student thinks, "This looks like a division problem, and since I can multiply both numbers by the same amount and get the same answer, I'll just multiply both numbers by 100. So, now I need to find 1575 ÷ 150 and this will give me the same answer. I did the division and got 10.5, which means that Victor can make 10 full ribbons, and he has enough left over to make half a ribbon."

In grade four, students identify prime numbers, composite numbers, and factor pairs. In grade six, students build on prior knowledge and find the greatest common factor (GCF) of two whole numbers less than or equal to 100 and find the least common multiple (LCM) of two whole numbers less than or equal to 12 (**M.6.7**). Teachers might employ compact methods for finding the LCM and GCF of two numbers, such as the ladder method discussed below and other methods.

Example: Division Using Single Digits Instead of Totals **M.6.7**

To find the LCM and GCF of 120 and 48, one can use the "ladder method" to systematically find common factors of 120 and 48 and identify the factors that 120 and 48 do not have in common. The GCF becomes the product of all those factors that 120 and 48 share, and the LCM is the product of the GCF and the remaining uncommon factors of 120 and 48.

Common Factors	Remaining Numbers	
3	120	48
4	40	16
2	10	4
	5	2

With the ladder method, common factors (3, 4, 2 in this case) are divided from the starting and remaining numbers until there are no more common factors to divide (5, 2). The GCF is then $3 \cdot 4 \cdot 2 = 24$, and the LCM is $24 \cdot 5 \cdot 2 = 240$.

Note: The grade six standard requires only that students find the GCF of numbers less than or equal to 100 and the LCM of numbers less than or equal to 12.

The Number System
Apply and extend previous understandings of numbers to the system of rational numbers. **M.6.8** Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation. **M.6.9** Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates. a. Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$, and that 0 is its own opposite. b. Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes. c. Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

In grade six, students begin the formal study of negative numbers, their relationship to positive numbers, and the meaning and uses of absolute value. Students use rational numbers (expressed as fractions, decimals, and integers) to represent real-world contexts and understand the meaning of 0 in each situation (**M.6.8**). Where the context gives rise to the basic meaning of $0 = (+n) + (-n)$, count models (e.g., positive or negative electric charge, credits and debits) will help develop student understanding of the relationship between a number and its opposite. In addition, measurement contexts such as temperature and elevation can contribute to student understanding of these ideas (**MHM1, MHM2, MHM4**).

Note that the standards do not specifically mention the set of integers (consisting of the whole numbers and their opposites) as a distinct set of numbers. Rather, the standards are focused on student understanding of the set of rational numbers in general (consisting of whole numbers, fractions, and their opposites). Thus, although early instruction in positive and negative numbers will likely start with examining whole numbers and their opposites, students must also work with negative fractions (and decimals) at this grade level. Ultimately, students learn that all numbers have an "opposite."

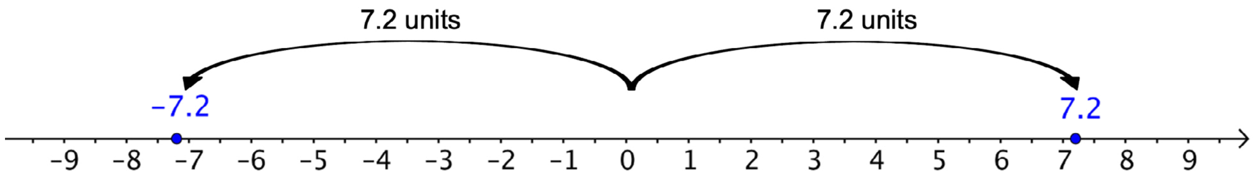
Examples of Rational Numbers in Context	M.6.8
1. All substances are made up of atoms, and atoms have protons and electrons. A proton has a positive charge, represented by "+1," and an electron has a negative charge, represented by "–1." A group of 5 protons has a total charge of +5, and a group of 8 electrons has a total charge of –8. One positive charge combines with one negative charge to result in a "neutral charge," which can be represented by (+1) + (–1) = 0. So, for example, a group of 4 protons and 4 electrons together would have a neutral charge, since there are 4 positive charges to combine with 4 negative charges. We could write this as (+4) + (–4) = 0. a. What is the overall charge of a group of 3 protons and 3 electrons? b. What is the overall charge of a group of 5 protons with no electrons? c. What is the overall charge of a group of 4 electrons with no protons?	
2. In a checking account, credits to the account are recorded as positive numbers (since they add money to the account), and debits to the account are recorded as negative numbers (since they take money away from the account). a. Explain the meaning of an account statement that shows a total balance of –$100.15. b. Explain the meaning of an account statement that shows a total balance of $225.78. c. If a person's bank statement shows –$45.67, then explain how he or she can get to a $0 balance.	
3. At any place on Earth, the elevation of the ground on which you are standing is measured by how far above or below the average level of water in the ocean (called sea level) the ground is. a. Discuss with a partner what an elevation of 0 means. Sketch a picture of what you think this means. b. Death Valley's Badwater Basin, located in California, is the point of lowest elevation in North America, at 282 feet below sea level. Explain why we would use a negative rational number to express this elevation. c. Mount Whitney is the highest mountain in California, at a height of 14,505 feet above sea level. Explain why we would use a positive number to express this elevation.	

In prior grades, students work with positive fractions, decimals, and whole numbers on the number line and in the first quadrant of the coordinate plane. In grade six, students extend the number line to represent all rational numbers, focusing on the relationship between a number and its opposite—namely, that they are equidistant from 0 on a number line (**M.6.9**). Number lines may be either horizontal or vertical (such as on a thermometer); experiencing both types will facilitate students' movement from number lines to coordinate grids.

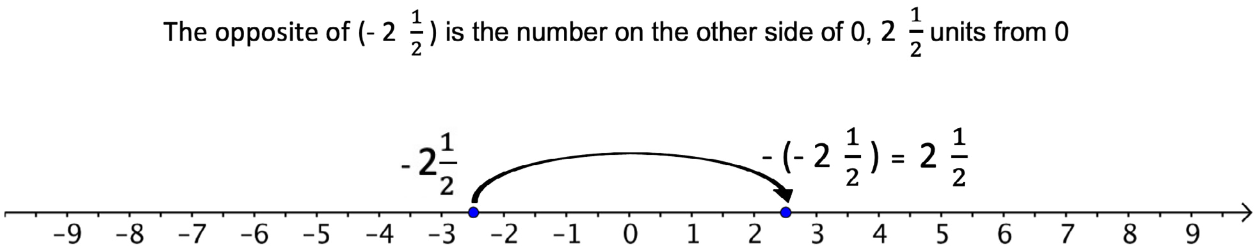
The Symbol "-" has multiple meanings.

The symbol "-" has several uses in mathematics. Since kindergarten, students have used this symbol to represent subtraction. In grade six they are responsible for understanding that the same symbol can be used to mean ***negative***, as in **-5**. (Negative numbers have also been represented with a "raised" minus sign, such as in **⁻5**; however, this practice is not consistent, and therefore teachers should use the more common minus sign.) However, students must also learn that the symbol "-" represents ***the opposite of***, as in, **-5** is the opposite of **5** since they are both the same distance from 0." This latter use is probably the most important, as it can be applied to cases such as, " - (-9) is the opposite of the opposite of 9, which is 9." When viewing a stand-alone expression such as **-k**, students might erroneously think the expression represents a negative number. However, if the value of **k** itself is a negative number (that is, if k = -3), then -k = -(-3) = 3. Thus, reading **–k** as "the opposite of **k**" is a more accurate way of reading this expression. Teachers should be consistent in using the word minus only when referring to subtraction and should use the word negative when referring to numbers like "-6" (that is, as opposed to saying "minus six").

In grade seven, students will explore operations with positive and negative rational numbers, so it is important that they develop a firm understanding of the relationship between positive and negative numbers and their opposites in grade six. Students recognize that a number and its opposite are the same distance from 0 on a number line, as in 7.2 and – 7.2 being the same distance from 0:



In addition, students recognize the symbol (-) as meaning the opposite of, and that in general, the opposite of a number is the number on the other side of 0 at the same distance from 0 as the original number. For example, - (- $2\frac{1}{2}$) is "the opposite of the opposite of $2\frac{1}{2}$" which is $2\frac{1}{2}$:



This understanding will help students develop the concept of absolute value, as the absolute value of a number is defined as its distance from 0 on a number line.

Students' previous work in the first quadrant grid helps them recognize the point where the x-axis and y-axis intersect as the origin. Grade six students identify the four quadrants and the appropriate quadrant for an ordered pair based on the signs of the coordinates (**M.6.9**). For example, students recognize that in Quadrant II, the signs of all ordered pairs would be (–, +). Students understand the relationship between two ordered pairs differing only by signs as reflections across one or both axes. For example, in the ordered pairs (–2, 4) and (–2, –4), the y-coordinates differ only by signs, which represents a reflection across the x-axis. A change in the x-coordinate from (–2, 4) to (2, 4), represents a reflection across the y-axis. When the signs of both coordinates change—for example, when (2, –4) changes to (–2, 4)—the ordered pair is reflected across both axes.

The Number System

Apply and extend previous understandings of numbers to the system of rational numbers.

M.6.10

Understand ordering and absolute value of rational numbers.

a. Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. (e.g., interpret –3 > –7 as a statement that –3 is located to the right of –7 on a number line oriented from left to right.)
b. Write, interpret, and explain statements of order for rational numbers in real-world contexts (e.g., write -3^{o} C > -7^{o} C to express the fact that -3^{o} C is warmer than -7^{o} C).
c. Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. (e.g., for an account balance of –30 dollars, write |–30| = 30 to describe the size of the debt in dollars).
d. Distinguish comparisons of absolute value from statements about order. (e.g., recognize that an account balance less than –30 dollars represents a debt greater than 30 dollars.)

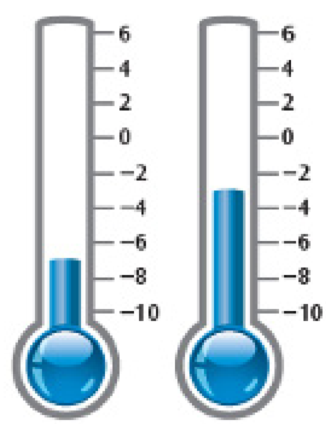
M.6.11

Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

In grade six, students reason about the order and absolute value of rational numbers (**M.6.10**) and solve real-world and mathematical problems by graphing in all four quadrants of the coordinate plane (**M.6.11**). Students use inequalities to express the relationship between two rational numbers. Working with number line models helps students internalize the order of the numbers — larger numbers on the right or top of the number line and smaller numbers to the left or bottom of the number line. Students correctly locate rational numbers on the number line, write inequalities, and explain the relationships between numbers. Students understand the absolute value of a rational number as its distance from zero on the number line and correctly use the absolute value symbol (e.g., |3| = 3, |-2| = 2). They distinguish comparisons of absolute value from statements about order (**M.6.10**).

Example: Comparing Rational Numbers **M.6.10b**

One of the thermometers at right shows -3^{o} C, and the other shows -7^{o} C. Which thermometer shows which temperature? Which is the colder temperature? How much colder? Write an inequality to show the relationship between the temperatures and explain how the model shows this relationship.



Solution:

On a vertical number line, negative numbers get "more negative" as we go down the line, so it appears that the thermometer on the left must read -7^{o} C and the thermometer on the right must read -3^{o} C. By counting spaces, the thermometer on the left reads a temperature colder by 4 degrees. Related inequalities are -7 < -3 and -3 > -7.

Common Misconceptions

With positive numbers, the absolute value (distance from zero) of the number and the value of the number are the same. However, students might be confused when they work with the absolute values of negative numbers. For negative numbers, as the value of the number decreases, the absolute value increases. For example, –24 is less than –14 because –24 is located to the left of –14 on the number line. However, the absolute value of –24 is greater than the absolute value of –14 because it is farther from zero. Students may also erroneously think that taking the absolute value means to "change the sign of a number," which is true for negative numbers but not for positive numbers or 0.

Adapted from ADE 2010, NCDPI 2013b, and KATM 2012, 6th Grade Flipbook

Domain: Expressions and Equations

A critical area of instruction in grade six is writing, interpreting, and using expressions and equations. In previous grades, students write numerical equations and simple equations involving one operation with a variable. In grade six, students start the systematic study of equations and inequalities and methods to solve them.

Students understand that mathematical expressions represent calculations with numbers. Some numbers, such as 2 or $\frac{3}{4}$, might be given explicitly. Other numbers are represented by letters, such as x, y, P, or n. The calculation represented by an expression might use a single operation, as in $4+3$ or $3x$, or a series of nested or parallel operations, as in $3(a+9)-\frac{b}{9}$. An expression may consist of a single number, even 0.

Students understand an equation as a statement that two expressions are equal. An important aspect of equations is that the two expressions on either side of the equal sign may not always be equal; that is, the equation might be a true statement for some values of the variable(s) and a false statement for others (adapted from UA Progressions Documents 2011d).

Expressions and Equations

Apply and extend previous understandings of arithmetic to algebraic expressions.

M.6.12

Write and evaluate numerical expressions involving whole-number exponents.

M.6.13

Write, read and evaluate expressions in which letters stand for numbers.

a. Write expressions that record operations with numbers and with letters standing for numbers. (e.g., Express the calculation, "Subtract y from 5" as $5 - y$.)
b. Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. (e.g., Describe the expression $2 (8 + 7)$ as a product of two factors; view $(8 + 7)$ as both a single entity and a sum of two terms.)
c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole number exponents, in the conventional order when there are no parentheses to specify a particular order: Order of Operations (e.g., use the formulas $V = s^3$ and $A = 6 s^2$ to find the volume and surface area of a cube with sides of length $s = 1/2$).

Students demonstrate understanding of the meaning of exponents by writing and evaluating numerical expressions with whole number exponents. The base can be a whole number, positive decimal, or a positive fraction (**M.6.12**). Students should work with a variety of expressions and problem situations to practice and deepen their skills. They can start with simple expressions to evaluate and move to more complex expressions. For example, they begin with simple whole numbers and move to fractions and decimal numbers (**MHM2, MHM6**).

Examples	**M.6.12**
• What is the side length of a cube with a volume 5^3 cubic cm? (Answer: 5 cm) • Write 10,000=10×10×10×10 with an exponent. (Answer: 10^4) • Andrea had half a pizza. She gave half of it to Marcus. Then Marcus gave half of what he had to Roger. Use exponents to write the amount of pizza Roger has. (Answer: $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=(\frac{1}{2})^3$)	
Evaluate the following: • 4^3 (Answer: $4 \times 4 \times 4 = 64$) • $5 + 2^4 \cdot 6$ (Answer: $5 + 2 \cdot 2 \cdot 2 \cdot 2 \cdot 6 = 5 + 16 \cdot 6 = 5 + 96 = 101$) • $7^2 - 24 \div 3 + 26$ (Answer: $7 \cdot 7 - 24 \div 3 + 26 = 49 - 24 \div 3 + 26 = 49 - 8 + 26 = 67$)	

Grade six marks a foundational year for building the bridge between concrete concepts of arithmetic and the abstract thinking of algebra. Visual representations and concrete models (such as algebra tiles, counters, and cubes) can help students develop understanding as they move toward using abstract symbolic representations.

Common Misconceptions

Students in grade six may not understand how to read the operations referenced with notations (e.g., x^3, $4x$, 3(x+2y), (a+3a)). Students are learning the following:

- x^3 means $x \cdot x \cdot x$, not $3x$ or 3 times x.
- 4x means 4 times x or $x+x+x+x$, not forty-something.
- When $4x$ is evaluated where $x=7$, substitution does not result in the expression meaning 47.
- For expressions like $a+3a$, students need to understand a as $1a$ to know that $a+3a=4a$ not $3a^2$.

The use of the "*x*" notation as both the variable and the operation of multiplication may also be a source of confusion for students. In addition, students may need an explanation for why $x^0=1$ for all non-zero numbers x. Full explanations of this and other rules of working with exponents appear in the document on grade eight.

Adapted from ADE 2010, NCDPI 2013b, and KATM 2012, 6th Grade Flipbook.

Students write, read, and evaluate expressions in which letters (called *variables*) stand for numbers (**M.6.13**). Grade six students write expressions that record operations with numbers and variables. Students need opportunities to read algebraic expressions to reinforce that the variable represents a number and therefore behaves according to the same rules for operations that numbers do (e.g., the distributive property).

Examples of Interpreting Expressions	**M.6.13a**

- Some (unknown) number plus 21 is represented by the expression $r+21$.
- Six (6) times some number n is represented by the expression $6 \cdot n$.
- The variable s divided by 4, as well as one-quarter of *s*, is represented by the expression $\frac{s}{4}$.
- The variable *r* minus 4.5, or 4.5 less than *r*, is represented by the expression $r-4.5$.
- Three (3) times the sum of a number and 5 is represented by the expression $3(x+5)$.

Adapted from NCDPI 2013.

The multiplication and division symbols × and ÷ are replaced by the conventions of algebraic notation as students move from numerical to algebraic work. Students learn to use either a dot for multiplication (e.g., $1 \cdot 2 \cdot 3$ instead of $1 \times 2 \times 3$) or simple juxtaposition (e.g., $3x$ instead of $3 \times x$), which is potentially confusing. During the transition, students may indicate all multiplications with a dot, writing $3 \cdot x$ for $3x$. Students also learn that $x \div 2$ can be written as $\frac{x}{2}$ (adapted from UA Progressions Documents 2011d).

Students identify the parts of an algebraic expression using mathematical vocabulary such as *variable*, *coefficient*, *constant*, *term*, *factor*, *sum*, *difference*, *product*, and *quotient* (**M.6.13b**). They should understand terms are the parts of a sum, and when a term is an explicit number, it is called a constant. When the term is a product of a number and a variable, the number is called the coefficient of the variable. Variables are letters that represent numbers. Development of this common mathematical vocabulary helps students understand the structure of expressions and explain their process for evaluating expressions.

Examples of Expression Language	**M.6.13b**

In the expression $x^2 + 5y + 3x + 6$, the variables are *x* and *y*.

- There are 4 terms: x^2, $5y$, $3x$, and 6.
- There are 3 variable terms: x^2, $5y$, and $3x$. These have coefficients of 1, 5, and 3, respectively.
- The coefficient of x^2 is 1, since $x^2=1 \cdot x^2$.
- The term 5y represents $y+y+y+y+y$ or $5 \cdot y$.
- There is one constant term: 6.
- The expression shows a sum of all four terms.

Adapted from NCDPI 2013.

Grade six students evaluate various expressions at specific values of their variables, including expressions that arise from formulas used in real-world problems. Examples, where students evaluate the same expression at several different values of a variable, are important for the later development of the concept of a function, and these should be experienced more frequently than problems wherein the values of the variables stay the same and the expression continues to change (**MHM1, MHM2, MHM3, MHM4, MHM6**).

Examples of Evaluating Expressions and Formulas **M.6.13c**

1. Evaluate the two expressions $5(n+3)+7n$ and $12n+15$ for $n = 0, \frac{1}{2}$, and 7.5. What do you notice?
2. The expression $c+0.07c$ can be used to find the total cost of an item with 7% sales tax added, where c is the cost of the item before taxes are added. Use this expression to find the total cost of an item that costs $25, then an item that costs $250, and finally an item that costs $25,000.
3. The perimeter of a parallelogram is found using the formula $P=2l+2w$. What is the perimeter of a rectangular picture frame with dimensions of 8.5 inches by 11 inches?

Adapted from NCDPI 2013.

Expressions and Equations

Apply and extend previous understandings of arithmetic to algebraic expressions.

M.6.14

Apply the properties of operations to generate equivalent expressions (e.g., apply the distributive property to the expression $3(2 + x)$ to produce the equivalent expression $6 + 3x$; apply the distributive property to the expression $24x + 18y$ to produce the equivalent expression $6(4x + 3y)$; apply properties of operations to $y + y + y$ to produce the equivalent expression $3y$).

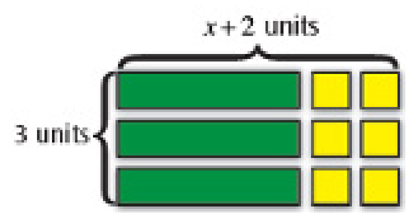
M.6.15

Identify when two expressions are equivalent; i.e., when the two expressions name the same number regardless of which value is substituted into them. (e.g., The expressions $y + y + y$ and $3y$ are equivalent because they name the same number regardless of which number y stands for.)

Students use their understanding of multiplication to interpret $3(2+x)$ as 3 groups of $(2+x)$ or the area of a rectangle of lengths 3 units and $(2+x)$ units (**MHM2, MHM3, MHM4, MHM6, MHM7**). They use a model to represent x and make an array to show the meaning of $3(2+x)$. They can explain why it makes sense that $3(2+x)$ is equal to $6+3x$. Manipulatives such as algebra tiles, which make use of the area model to represent quantities, may be used to show why this is true. Note that with algebra tiles, a 1-by-1 square represents a unit (the number 1), while the variable x is represented by a rectangle of dimensions 1 by x (that is, the longer side of the x-tile is not commensurate with a whole number of unit tiles, and therefore it represents an unknown length).

Example of Basic Reasoning with Algebra Tiles **M.6.14**

Students can recognize $3(x+2)$ as representing the area of a rectangle of lengths 3 units and $(x+2)$ units. Using the appropriate number of tiles (or a sketch), students can see that there are $3\cdot2=6$ and $3\cdot x=3x$ units altogether, so that $3(x+2) = 3x+6$.



Adapted from NCDPI 2013.

Standards **M.6.14-M.6.15** highlight the importance of understanding the distributive property, which is the basis for combining like terms in an expression or equation. For instance, students understand that $4a+7a=11a$, because

$$4a+7a$$
$$= (4+7)a$$
$$= 11a.$$

It is important for students to develop the ability to use the distributive property flexibly — for example, to see that $3(2x+5)$ is the same as $(2x+5)3$ and $6x+15$. Students generate equivalent expressions using the associative, commutative, and distributive properties and can prove the expressions are equivalent (**MHM1, MHM2, MHM3, MHM4, MHM6**).

Example: Equivalent Expressions	**M.6.15**
Show that the two expressions $5(n+3)+7n$ and $12n+15$ are equivalent. ***Solution:*** "By applying the distributive property, I know that $5(n+3)+7n$ can be rewritten as $5n+15+7n$. Also, since $5n+7n=(5+7)n=12n$, I can write the expression as $12n+15$."	

Expressions and Equations

Reason about and solve one-variable equations and inequalities.

M.6.16
Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

M.6.17
Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number or depending on the purpose at hand, any number in a specified set.

M.6.18
Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all nonnegative rational numbers.

M.6.19
Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams.

In previous grade levels, students explore the concept of equality. In grade six, students explore equations as one expression being set equal to a specific value. A solution is a value of the variable that makes the equation true. Students use various processes to identify such values that, when substituted for the variable, will make the equation true (**M.6.16**). Students can use manipulatives and pictures (e.g., tape diagrams) to represent equations and their solution strategies. When writing equations, students learn to be precise in their definition of a variable — for example, writing "*n* equals John's age in years" as opposed to writing only that "*n* is John" (**M.6.17**) [**MHM6**].

Students have been working with manipulatives and pictures (e.g., tape diagrams) since their introduction to problem solving in their early elementary grades. Tape diagrams were introduced in grades two and three. As a result, using students' skills and understandings of tape diagrams can serve to make the transition from tape diagrams to equations a natural progression.

Examples: Modeling and Solving Equations of the Form p+x=q and px=q	**M.6.17-M.6.18**

Joey had 26 game cards. His friend Richard gave him some more, and now Joey has 100 cards. How many cards did Richard give to Joey? Write an equation and solve your equation.

Solution:
Since Richard gave him some more cards, we let n represent the number of cards that Richard gave Joey. This means he now has $26+n$ cards. But the number of cards Joey has is 100, so we get the equation $26+n=100$. Using the relationship between addition and subtraction, we see that $n=100-26=74$, which means that his friend gave him 74 cards. This equation can be represented with a tape diagram:

100	
26	n

2. A book of tickets for rides at an amusement park costs $30.00. Each ticket costs $2.50. How many tickets come in each book? Write and solve an equation that represents this situation.

   Solution:
   If s represents the number of tickets in one booklet, then $(2.50)s$ is the cost of tickets in dollars. Since the cost of one book is $30.00, solving the equation $(2.50)s=30.00$ would result in the number of tickets. To solve this equation, we realize that if $2.50 \times s=30.00$, then $s=30.00 \div 2.50=12$. This means there are 12 tickets in each book.

3. Meagan spent $56.58 on three pairs of jeans. If each pair of jeans costs the same amount, write an equation that represents this situation and solve it to determine the price of one pair of jeans.

$56.58		
J	J	J

   Solution:
   If J represents the cost of one pair of jeans in dollars, then the equation becomes $3J=56.58$. If we solve this for J, we find $J=56.58 \div 3=18.86$. This means each pair of jeans cost $18.86.

4. Julio was paid $20.00 for babysitting. He spent $1.99 on a package of trading cards and $6.50 on lunch. Write and solve an equation to show how much money Julio has left.

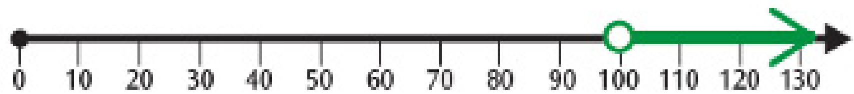
   Solution:
   One equation might be $1.99+6.50+x=20.00$, where x represents the amount of money (in dollars) that Julio has left. We find that $x=11.51$, so Julio has $11.51 left.

Many real-world situations are represented by inequalities. In grade six, students write simple inequalities involving < or > to represent real-world and mathematical situations, and they use the number line to model the solutions (**M.6.19**). Students learn that when representing inequalities of these forms on a number line, the common practice is to draw an arrow on or above the number line with an open circle on or above the number in the inequality. The arrow indicates the numbers greater than or less than the number in question, and the solutions extend indefinitely. The arrow is a solid line indicating that even fractional and decimal amounts (i.e., points between marked values on the line) are included in the solution set.

Examples: Inequalities of the Form $x<c$ and $x>c$ **M.6.19**

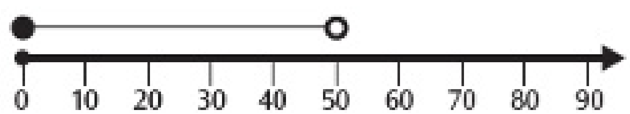
1. A class must raise more than $100 to go on a field trip. Let m represent the amount of money in dollars that the class raises. Write an inequality that describes how much the class needs to raise. Represent this inequality on a number line.

   Solution:
   Since the amount of money, m, needs to be greater than 100, the inequality is $m>100$. A number line diagram for this might look like this:



Students may think that since the amount of money m must be greater than 100 and 100 cannot be in the solution set. An open circle is used to denote that a number is not in the solution set and the arrow indicates that there is an infinite number of possible solutions.

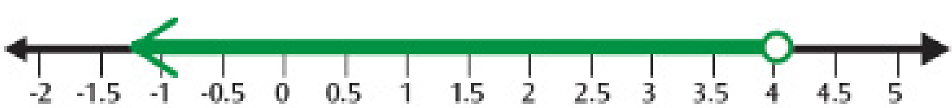
2. The Flores family spent less than $50 on groceries last week. Write an inequality that describes this situation, and graph the solution on a number line.
   Solution:
   If we let g represent the amount of money (in dollars) that the family spent on groceries last week, then the inequality becomes $g<50$. We might represent this in the following way:



In this example, the Flores family could not have spent a negative amount of money on groceries, so the arrow would stop precisely at $0; typically, this would be represented with a dot over 0 rather than the arrow.

3. Graph $x<4$.

Solution: This graph represents all numbers less than 4:



Expressions and Equations
Represent and analyze quantitative relationships between dependent and independent variables. **M.6.20** Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. (e.g., In a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation d = 65*t* to represent the relationship between distance and time.)

In grade six, students investigate the relationship between two variables, beginning with the distinction between independent and dependent variables (**M.6.20**). The **independent variable** is the variable that can be changed; the **dependent variable** is the variable that is affected by the change in the independent variable. Students recognize that the independent variable is graphed on the *x*-axis and the dependent variable is graphed on the *y*-axis. They also understand that not all data should be graphed with a line. Discrete data would be graphed only with coordinates.

Students show relationships between quantities with multiple representations, using language, a table, an equation, or a graph. Translating between multiple representations helps students understand that each form represents the same relationship and provides a different perspective on the relationship.

Example: Exploring Independent and Dependent Variables **M.6.20**

Stephanie is helping her band raise money to fund a field trip. The band decided to sell school pennants, which come in boxes of 20. Each pennant sells for $1.50. The partially completed table at right shows money collected for different numbers of boxes sold.

a. Complete the table for the remaining values of *m*.
b. Write an equation for the amount of money, *m*, that will be collected if *b* boxes of pennants are sold. Which is the independent variable and which is the dependent variable?
c. Graph the relationship by using ordered pairs from the table.
d. Calculate how much money will be collected if 100 boxes of school pennants are sold.

Boxes Sold (*b*)	Money Collected (*m*)
1	$30
2	
3	
4	
5	$150
6	
7	
8	

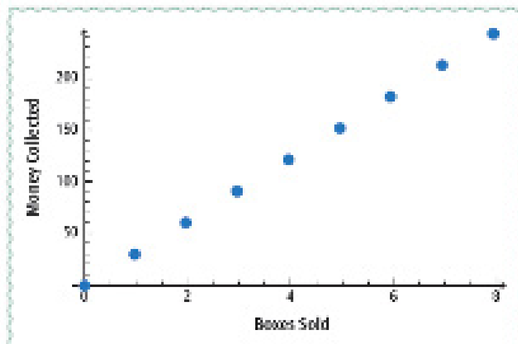
a.

Boxes Sold (b)	Money Collected (m)
1	\$30
2	
3	
4	
5	\$150
6	
7	
8	

Boxes Sold (b)	Money Collected (m)
1	\$30
2	\$60
3	\$90
4	\$120
5	\$150
6	\$180
7	\$210
8	\$240

b. Students may derive the equation $m=30b$, representing the fact that when b boxes are sold at \$30 per box, then the total amount of money collected is $30b$, dollars. In this case, the independent variable is the number of boxes sold, b, and the money collected is the dependent variable. This equation certainly is a valid way to make sense of the problem, in that the amount of money collected depends on the number of boxes sold. However, if one has fund-raising goals, then it would be natural to think of the relationship as $b = \frac{m}{30}$, in the sense that the number of boxes needed to be sold depends on the fundraising target.

c. If we graph the relationship as (b,m), then we obtain the graph shown, which illustrates the relationship $m=30b$. (In grade seven, students will more fully explore graphs of proportional relationships such as this one.)



d. Using the equation derived in solution b, $m=30b$, we use 100 for the value of b and find the amount of money collected will be \$3,000.

Domain: Geometry

In grade six, students extend their understanding of length, area, and volume as they solve problems by applying formulas for the area of triangles and parallelograms and volume of rectangular prisms.

Geometry
Solve real-world and mathematical problems involving area, surface area, and volume. **M.6.21** Find the area of right triangles, other triangles, special quadrilaterals and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems. **M.6.22** Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas V = l w h and V = B h to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems. **M.6.23** Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems. **M.6.24** Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

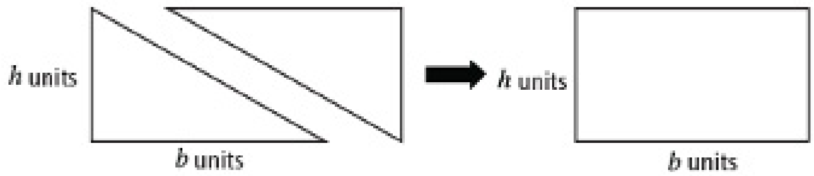
Sixth grade students build on their work with area from previous grade levels by reasoning about relationships among shapes to determine area, surface area, and volume. Students in grade six continue to understand area as the number of squares needed to cover a plane figure. They find areas of right triangles, other triangles, and special quadrilaterals by decomposing these shapes, rearranging or removing pieces, and relating the shapes to rectangles. As students compose and decompose shapes to determine areas, they learn that area is conserved when composing or decomposing shapes. For example, students will decompose trapezoids into triangles and/or rectangles and use this reasoning to find formulas for the area of a trapezoid. Students know area formulas for triangles and some special quadrilaterals, in the sense of having an understanding of why the formula works and how the formula relates to the measure (area) and the figure (**M.6.21**).

Prior to being exposed to the formulas for areas of different shapes, students can find areas of shapes on centimeter grid paper by duplicating, composing, and decomposing shapes. These experiences will familiarize students with the processes that result in the derivations of the following area formulas.

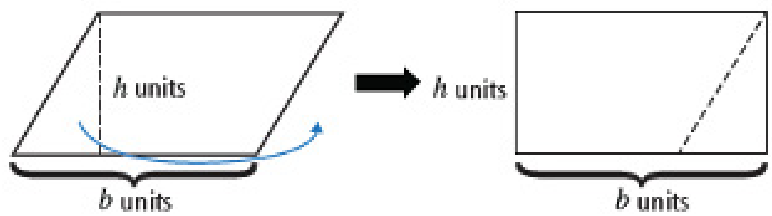
Example: Deriving Area Formulas **M.6.21 (MHM3, MHM7)**

Starting with a basic understanding that the area of a rectangle of base b units and height h units is bh square units, along with the relationship between rectangles and triangles and the law of conservation of area, students can justify area formulas for various shapes.

Right Triangles: Since two right triangles of base b and height h can be composed to form a rectangle of the same base and height, the triangle must have an area half that of the rectangle. Thus, the area of a right triangle of base b and height h is $\frac{1}{2}bh$ square units.

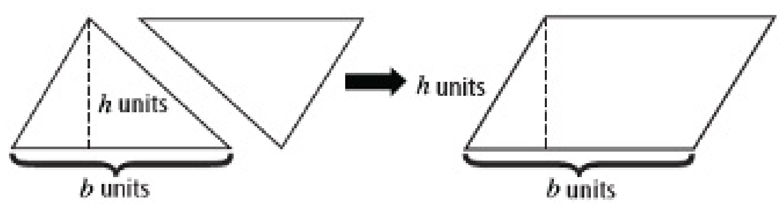


Parallelograms: If we define the height of the parallelogram to be the length of a perpendicular segment from base to base, then a parallelogram of base b and height h has the same area (bh square units) as a rectangle of the same dimensions. We cut off a right triangle as shown and move it to complete the rectangle.



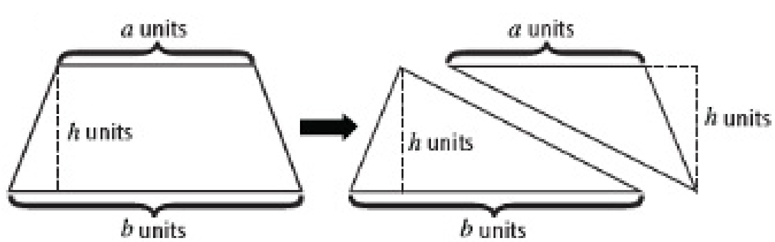
Non-Right Triangles:

Non-right triangles of base b units and height h units can now be duplicated to make parallelograms. By similar reasoning used with right triangles and rectangles, the area of such a triangle is $\frac{1}{2}bh$ square units. (One can show the same holds true for obtuse triangles.)



Trapezoids:

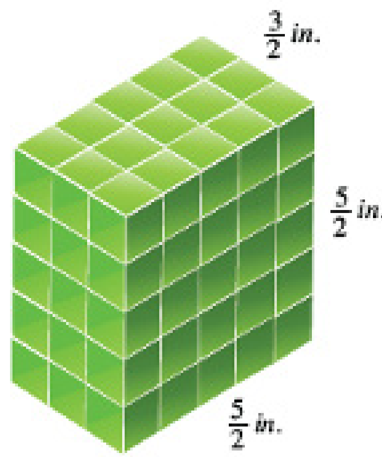
Trapezoids can be deconstructed into two triangles of bases a and b, showing that the area of a trapezoid can be found by $\frac{1}{2}ah + \frac{1}{2}bh = \frac{1}{2}(a+b)h$.



Previously, students calculated the volume of right rectangular prisms using whole number edges and understood doing so as finding the number of unit cubes (i.e., 1 ×1 ×1 cubic unit) within a solid shape. In grade six, students extend this work to unit cubes with fractional edge lengths. For example, they determine volumes by finding the number of unit cubes of dimensions $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$ within a figure with fractional side lengths. Students draw diagrams to represent fractional side lengths, and in doing so they connect finding these volumes with multiplication of fractions (**M.6.22**).

Example: Counting Fractional Cubic Units **M.6.22**

The model at right shows a rectangular prism with dimensions $\frac{3}{2}$ inches, $\frac{5}{2}$ inches, and $\frac{5}{2}$ inches. Each of the cubic units shown in the model has a volume of $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ cubic inches. Students should reason why each of these units has this volume (i.e., by discovering that 8 of them fit in a 1×1×1 cube). Furthermore, students explain why the volume of the rectangular prism is given by $\frac{3}{2} \times \frac{5}{2} \times \frac{5}{2}$ cubic inches and why the volume can also be determined by finding $(3 \times 5 \times 5) \times (\frac{1}{8}$ cubic inch).

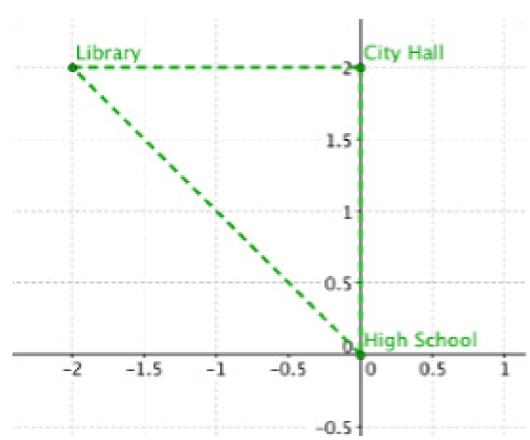


When students find areas, surface areas, and volumes, modeling opportunities are presented (**MHM4**), and students must attend to precision with the types of units involved (**MHM6**). Standard **M.6.23** calls for students to represent shapes in the coordinate plane. Students find lengths of sides that contain vertices with a common *x*- or *y*-coordinate, representing an important foundational step toward grade-eight understanding of how to use the distance formula to find the distance between any two points in the plane. In addition, grade six students construct three-dimensional shapes using nets and build on their work with area (**M.6.24**) by finding surface areas with nets.

Example: Polygons in the Coordinate Plane **M.6.23**

On a grid map, the library is located at (–2, 2), the city hall building is located at (0, 2), and the high school is located at (0, 0).

a. Represent the locations as points on a coordinate grid with a unit of 1 mile.
b. What is the distance from the library to the city hall building?
c. What is the distance from the city hall building to the high school? How do you know?
d. What is the shape that results from connecting the three locations with line segments?
e. The city council is planning to place a city park in this area. What is the area of the planned park?



Instructional Focus
The standards in the cluster “Solve real-world and mathematical problems involving area, surface area, and volume” regarding areas of triangles and volumes of right rectangular prisms (**M.6.21-M.6.22**) connect to major work in the Expressions and Equations domain (**M.6.12-M.6.20**). In addition, standard **M.6.23** asks students to draw polygons in the coordinate plane, which supports major work with the coordinate plane in the Number System domain (**M.6.11**).

Domain: Statistics and Probability

A critical area of instruction in grade six is developing understanding of statistical thinking. Students build on their knowledge and experiences in data analysis as they work with statistical variability and represent and analyze data distributions. They continue to think statistically, viewing statistical reasoning as a four-step investigative process (UA Progressions Documents 2011e):

- Formulate questions that can be answered with data.
- Design and use a plan to collect relevant data.
- Analyze the data with appropriate methods.
- Interpret results and draw valid conclusions from the data that relate to the questions posed.

Statistics and Probability
Develop understanding of statistical variability.
M.6.25 Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers. (e.g., “How old am I?” is not a statistical question, but “How old are the students in my school?” is a statistical question because one anticipates variability in students’ ages.)
M.6.26 Through informal observation, understand that a set of data collected to answer a statistical question has a distribution which can be described by its center (mean/median), spread (range), and overall shape.
M.6.27 Recognize that a measure of center for a numerical data set summarizes all of its values with a single number.

Statistical investigations start with questions, which can result in a narrow or wide range of numerical values and ultimately result in some degree of variability (**M.6.25**). For example, asking classmates “How old are the students in my class in years?” will result in less variability than asking “How old are the students in my class in months?” Students understand that questions need to specifically demand measurable answers. For example, if a student wants to know about the exercise habits of fellow students at her school, a statistical question for her study could be “On average, how many hours per week do students at my school exercise?” This is much more specific than asking “Do you exercise?” Grade six students design survey questions that anticipate variability in the responses (ADE 2010, NCDPI 2013b, and KATM 2012, 6th Grade Flipbook).

One focus of grade six is the characterization of data distributions by measures of center and spread. To be useful, center and spread must have well-defined numerical descriptions that are commonly understood by those using the results of a statistical investigation (UA Progressions Documents 2011e). Grade six students analyze the center, spread, and overall shape of a set of data (**M.6.26**). As students analyze and/or compare data sets, they consider the context in which the data are collected and identify clusters, peaks, gaps, and symmetry in the data. Students learn that data sets contain many numerical values that can be summarized by one number, such as a measure of center (mean and median) and range.

Describing Data	**M.6.27**
A *measure of center* gives a numerical value to represent the central tendency of the data (e.g., midpoint of an ordered list [median] or the balancing point). The *range* provides a single number that describes how widely the values vary across the data set. Another characteristic of a data set is the measure of *variability* (or spread from center) of the values.	

Measures of Center

Given a set of data values, students summarize the measure of center with the median or mean (**M.6.26**). The **median** is the value in the middle of an ordered list of data. This value means that 50 percent of the data is greater than or equal to it and that 50 percent of the data is less than or equal to it. When there is an even number of data values, the median is the arithmetic average of the two values in the middle.

The ***mean*** is the arithmetic average: the sum of the values in a data set divided by the number of data values in the set. The mean measures center in the sense that it is the hypothetical value that each data point would equal if the total of the data values were redistributed equally. Students can develop an understanding of what the mean represents by redistributing data sets to be level or fair (creating an equal distribution) and by observing that the total distance of the data values above the mean is equal to the total distance of the data values below the mean (reflecting the idea of a balance point).

Example: Exploring Independent and Dependent Variables — M.6.27

Consider the data shown in the following line plot of the scores for organization skills for a group of students.

a. How many students are represented in the data set?
b. What are the mean and median of the data set? Compare the mean and median.
c. What is the range of the data? What does this value tell you?

Solution:

a. Since there are 19 data points (represented by Xs) in the set, there are 19 students represented.
b. The mean of the data set can be found by adding all of the data values (scores) and dividing by 19. The calculation below is recorded as (score) × (number of students with that score):

$$\frac{0(1)+1(1)+2(2)+3(6)+4(4)+5(3)+6(2)}{19}=\frac{66}{19}\approx 3.47$$

From the line plot, the median of the data set appears to be 3. To check this, we can line up the data values in ascending order and look for the center:

$$0,\ 1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 6,\ 6$$

The median is indeed 3, since there are 9 data values to the left of the circled 3 and 9 values to the right of it. The mean is greater than the median, which makes sense because the data are slightly skewed to the right.

c. The range of the data is 6, which coincides with the range of possible scores.

Adapted from ADE 2010; KATM 2012, 6th Grade Flipbook; and NCDPI 2013.

Measures of Variability

In grade six, variability is measured by using the interquartile range or the mean absolute deviation. The **interquartile range (IQR)** describes the variability within the middle 50% of a data set. It is found by subtracting the lower quartile from the upper quartile. In a **box plot**, it represents the length of the box and is not affected by **outliers**. Students find the IQR of a data set by finding the upper and lower quartiles and taking the difference or by reading a box plot.

Statistics and Probability
Summarize and describe distributions.
M.6.28 Display numerical data in plots on a number line, including dot plots, histograms and box plots.
M.6.29 Summarize numerical data sets in relation to their context, such as by: a. Reporting the number of observations. b. Describing the nature of the attribute under investigation, including how it was measured and its units of measurement. c. Giving quantitative measures of center (median and/or mean), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered. d. Relating the choice of measures of center to the shape of the data distribution and the context in which the data were gathered.

Students in grade six use number lines, dot plots, histograms, and box plot graphs (**M.6.28**) to display data graphically. Students learn to determine the appropriate graph for displaying data and how to read data from graphs generated by others.

Graphical Displays of Data in Grade Six	M.6.28

- ***Dot plots*** are simple plots on a number line where each dot represents a piece of data in a data set. Dot plots are suitable for small to moderately sized data sets and are useful for highlighting the distribution of the data, including clusters, gaps, and outliers.

- A ***histogram*** shows the distribution of data using intervals on the number line. The height of each bar represents the number of data values in that interval. In most real data sets, there is a large amount of data, and many numbers will be unique.

- A ***box plot*** shows the distribution of values in a data set by dividing the set into quartiles. It can be graphed either vertically or horizontally. The box plot is constructed from the five-number summary (minimum, lower quartile, median, upper quartile, and maximum). These values give a summary of the shape of a distribution. Box plots display the degree of spread of the data and the skewness of the data and can help students compare two data sets.

Students in grade six interpret data displays and determine measures of center and variability from them. They summarize numerical data sets in relation to the context of the data (**M.6.29**).

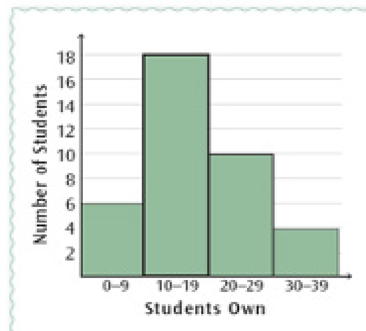
Example: Interpreting Data Displays **M.6.29**

1. Students in grade six were collecting data for a project in math class. They decided to survey the other two grade six classes to determine how many video games each student owns. A total of 38 students were surveyed. The data are shown in the table below, in no particular order. Create a data display. What are some observations that can be made from the data display?

11	21	5	12	10	31	19	13	23	33
10	11	25	14	34	15	14	29	8	5
22	26	23	12	27	4	25	15	7	
2	19	12	39	17	16	15	28	16	

Solution:

Students might make a histogram with 4 ranges (0–9, 10–19, 20–29, 30–39) to display the data. It appears from the histogram that the mean and median are somewhere between 10 and 19, since the data of so many students lie in this range. Relatively few students own more than 30 video games; in fact, further analysis may prove the data point 39 to be an outlier.



2. Ms. Wheeler asked each student in her class to write his or her age, in months, on a sticky note. The 28 students in the class brought their sticky notes to the front of the room and posted them in order on the whiteboard. The data set is listed below, in order from least to greatest. Create a data display. What are some observations that can be made from the data display?

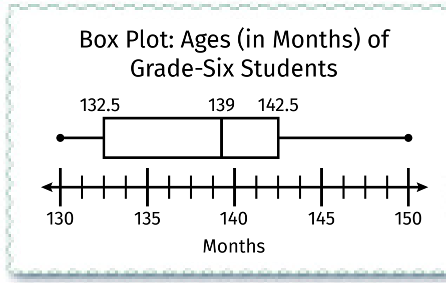
130	130	131	131	132	132	132	133	134	136
137	137	138	139	139	139	140	141	142	142
142	143	143	144	145	147	149	150		

Solution:

By finding the **five-number summary** of the data, we can create a box plot. The minimum data value is 130 months, the maximum is 150 months, and the median is 139 months. To find the first quartile (Q_1) and third quartile (Q_3), we find the middle of the upper and lower 50%. Since there is an even number of data points in each of these parts, we must find the average, so that $Q_1 = \frac{(132 + 133)}{2} = 132.5$ and $Q_3 = \frac{(142 + 143)}{2} = 142.5$. Thus, the five-number summary is as follows:

Minimum	First Quartile (Q_1)	Median	Third Quartile (Q_3)	Maximum
130	132.5	139	142.5	150

Now a box plot is easy to construct. The box plot helps to show that the middle 50% of values lie between 132.5 months and 142.5 months. Additionally, only 25% of the values are between 130 months and 132.5 months, and only 25% of the values are between 142.5 and 150.



Adapted from ADE 2010, NCDPI 2013b, and KATM 2012, 6th Grade Flipbook.

Instructional Focus

As students display and summarize numerical data (**M.6.28-M.6.29**), they strengthen mathematical practices such as making sense of given data (**MHM1**), using appropriate statistical models and measures (**MHM4, MHM5**), and attending to precision in calculating and applying statistical measures (**MHM6**).

Students can use online tools such as the following to create data displays:

- Box Plot Generator (*https://www.desmos.com/calculator/h9icuu58wn*)
- Histogram Creator (*https://www.desmos.com/calculator/lcnnbqh0nc*)

Essential Learning for the Next Grade

In grades six through eight, multiplication and division develop into powerful forms of ratio and proportional reasoning. The properties of operations take on prominence as students move from arithmetic to algebra. The theme of quantitative relationships also become explicit in grades six through eight, developing the formal concept of a function by grade eight. In addition, the foundations of deductive geometry are laid. The gradual development of data representations in kindergarten through grade five leads to the study of statistics in grades six through eight: evaluation of shape, center, and spread of data distributions; possible associations between two variables; and the use of sampling in making statistical decisions.

Guidance on Course Placement and Sequences

The West Virginia College- and Career-Readiness Standards for Mathematics for grades six through eight are comprehensive, rigorous, and non-redundant. Instruction in an accelerated sequence of courses will require compaction — not the former strategy of deletion. Therefore, careful consideration needs to be made before placing a student in higher mathematics course work in grades six through eight. Acceleration may get students to advanced course work, but it may lead to gaps in students' mathematical background.

Modeled after the Mathematics Framework for California Public Schools, Kindergarten Through Grade Twelve.

W. Clayton Burch
West Virginia Superintendent of Schools