WEST VIRGINIA
DEPARTMENT OF EDUCATION

MATHEMATICS

GRADE 6

Fermi Problems

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Task Title: Fermi Problems
Grade or Content Area: 6th Grade
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Original Task Creator: Illustrative Math
Quarter: 2

Rationale for Lesson and Associated Tasks

In this three-part math task, students use their knowledge of ratios and ratio reasoning to solve an unfamiliar, Fermi-style problem. A Fermi problem is an estimation problem designed to teach approximation of extreme calculations. The classic Fermi problem is, "How many piano tuners are there in Chicago?" Students might be more interested in, "How many sheets of paper would be in a stack that reaches from the floor to the ceiling of our classroom?"

Lesson and Associated Tasks Overview

Task: (click here)

In Part 1 of the task students reason informally about ratios and proportions through a hot cocoa problem. This task uses the type of reasoning that will be useful in Parts 2 and 3.

In Part 2, students are introduced to the type of thinking useful for Fermi problems. They will work as a class to break a Fermi problem into smaller questions that can be measured, estimated, or calculated. Using the answers to these smaller questions, students will use ratio reasoning to solve the problem.

In Part 3, students develop their own Fermi problem and solve it with the aim of promoting ratio reasoning and critical thinking skills developed in this unit.

Use discretion as to the amount of time given for each phase of the task. *Illustrative Mathematics* suggests that this is a one-day lesson. However, if students are focused on the task at hand and need additional time, the task can easily be split into a two-day lesson. In this document, it is written as a two-day lesson.

West Virginia College- and Career-Readiness State Standard

M.6.3

Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

a. Make tables of equivalent ratios relating quantities with whole number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.
b. Solve unit rate problems including those involving unit pricing and constant speed. (e.g., If it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?)
c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.
d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

Mathematical Habits of Mind (MHM)

MHM2. Reason abstractly and quantitatively.
MHM7. Look for and make use of structure.
MHM8. Look for and express regularity in repeated reasoning.

Mathematics Teaching Practices to Support Student Growth

2- Use and connect mathematical representations.
5- Pose purposeful questions.
6- Build procedural fluency from conceptual understanding.

Essential Understandings

- An understanding of ratios and proportional relationships, with an ability to apply that knowledge, is valuable in solving more complicated problems.
- An ability to break complex problems down into simpler parts is important in solving a real-world problem about ratios and rates.

Set-up Phase

1. Become an Expert in the Task

- Read through and work the Hot Cocoa problem. Identify questions your students may have or areas in which they may struggle. Working through the task allows the educator to find what students are being asked to do and gives the educator insight into the different ways students may approach the task.
- Prior to beginning this lesson with students, the teacher should research and become familiar with Fermi problems, thinking in particular about ones that might be appropriate for sixth-grade students.

2. Establish Small Groups

Students will work with a partner or in groups of three for this lesson.
It is important to establish standards for group work that can be used to help students stay focused during tasks and group work assignments. Standards can be developed by students, posted in the classroom, and referenced prior to group task activities.

Skill levels, leadership skills, and personalities are all considered when creating small (e.g. two/three students) groups. Small group collaboration works best when students have been provided previous opportunities to work together on a regular basis. Through weeks of teacher observation and making note of leadership skills, personalities, the ability to take criticism, to question, and think deeply about a task or problem will prove to be extremely helpful when creating small groups for this lesson and associated tasks. When creating the groups, all these factors help to eliminate the potential situation in which one student takes the lead and makes the decisions for the group. In this scenario, one student is gaining all the benefits of the task, while others do not. If a student is not engaged in conversation, this lesson and associated tasks will not be beneficial in helping all students to have meaningful discussions about the mathematics involved nor in analyzing the relationships inherent to the tasks. Students will be working together when placed in small (e.g. two or three students) to complete this task. The goal is for all students to participate in discussion and decision-making.

3. Develop Open-Ended Questions

Teachers should create a list of open-ended questions designed to support and scaffold the task for struggling students. These questions should purposefully direct students towards necessary decisions and/or provided information, previously learned content, and similarities and differences in their work versus that of group members. Some questions might include:

- What does the word proportional mean? Can you give an example of ratios that have a proportional relationship?

- When creating new proportional ratios, what are some of the mathematical methods you can use?
- Can you find numbers that would be easy to use to scale up and down?
- Can you draw a picture or diagram that would help you?
- What information do you know?
- What do you know about the ratio of cocoa to milk?
- What do you have to know before you can find…?

4. Gather Materials

- Computer and Presentation Device
- Copies of task documents from Illustrative Mathematics: (click here)
  The teacher may want to make two copies of the graphic organizer for each student, one for the class Fermi problem and one for the group problem.
- Cocoa powder & milk for demonstration (optional)
- Supplies for students to explore their Fermi problems (supplies needed will vary based on what students choose to explore). This may include devices or books in which to locate facts associated with their problem.

5. Anticipated Common Student Misconceptions

Teachers should create a list of possible student misconceptions that may occur. Anticipated misconceptions/errors:

- When creating proportional relationships, students may add or subtract the same number from each part of the ratio instead of scaling up or down through multiplication or division. Prompt students to look in notes (if available) or ask group members for ideas on how to correctly create proportional ratios. Be particularly careful in this situation, as students might say, "Add another cup of milk and another tablespoon of cocoa." Be sure to refer back to the ratio.
- Students may have trouble thinking logically about the steps needed to solve their Fermi problem.
- Students may calculate with large numbers and make simple arithmetic errors when solving their Fermi problems. Encourage students to round and estimate in order to determine if their answers seem logical in terms of the problem.

6. Plan for Early Finishers

Teachers should also plan lesson extensions for students who quickly complete the task.

- Encourage students to extend their Fermi problem on a larger scale. For example, a Fermi problem such as, "How far can all the students at a school reach if they stand fingertip to fingertip?" can be expanded to "How far can all of the students in West Virginia reach if they stand fingertip to fingertip?" or "How many students would be needed to reach from our school to the state capitol building, the nearest grocery store, the next town, etc.?"

Prior Instruction/Knowledge:

In Grade 6, ratios, rate, and percentage are connected to whole-number multiplication and division and the concepts of ratio and rate are used to solve problems.

- Prior to this lesson, students should have developed skills and understandings about ratio relationships, ways to write ratios, and how they are used to represent comparisons in real-world scenarios.

- Students should have developed skills and understandings about a variety of methods to determine equivalent ratios and understand them as proportional relationships.
- Student familiarity with different ways to show equivalent ratios, including pictorial, proportionally, and with a double number line, will allow them to appreciate that, regardless of how they are shown, the ratio relationship remains the same.

This lesson should be a culminating lesson in a unit on ratio.

Please review the following:
Educators Guide for Mathematics: Grade 6 (pages 4-17, pdf pages 2-15) (click here)

Prerequisite Skills

- Construct or complete a table of values to solve problems associated with a given relationship.
- Write a ratio or rate to compare two quantities.
- Given a proportional relationship represented by tables, graphs, models, or algebraic or verbal descriptions, identify the unit rate (constant of proportionality).

Supporting Skills

- Solve word problems using patterns.
- Calculate unit rates in number and word problems, including comparison of unit rates.
- Solve number and word problems using percent proportion, percent equation, or ratios.
- Relate a percent to its equivalent fraction or decimal.
- Write a proportion to model a word problem; solve proportions.
- Write a linear equation or inequality to represent a given number or word problem; solve.
- Model the concept of percent and relate to the value in decimal or fractional form.
- Convert measures of length, area, capacity, weight, and time expressed in a given unit to other units in the same measurement system in number and word problems.
- Determine the ratio or rate of change of a relation given a table or graph.

Impending Skills

- Generate a set of ordered pairs using a rule that is stated in verbal, algebraic, or table form; generate a sequence given a rule in verbal or algebraic form.
- Write equations to represent direct variation and use direct variation to solve number and word problems.
- Determine and use scale factors to reduce and enlarge drawings on grids to produce dilations.
- Use proportional reasoning to solve problems related to similar polygons.
- Calculate or estimate the percent of a number including discounts, taxes, commissions, and simple interest.
- Use dimensional analysis to rename quantities or rates.
- Calculate unit rates of ratios that include fractions to make comparisons in number and word problems.

Source: *The Quantile Framework for Mathematics*
https://metametricsinc.com/educators/quantile-for-educators/

Implementation Phase

Prior to Day 1 of this lesson, the teacher might introduce Fermi Problems to the class and have them compile a list of questions. Then each pair of students could make a “Top 5” list to turn in for teacher approval. In order to encourage ratio reasoning and the use of tools like double number lines and ratio tables, the teacher should look for problems that involve two quantities such as, “How long would it take to walk across the United States?” where time and distance are being compared. This will also allow the teacher to obtain any materials the students may need in order to solve their Fermi problem.

Day 1

- Provide students with a copy of the task.
- Introduce the Hot Cocoa problem.
- Use a Think/Pair/Share protocol to work through and discuss the problem.
- Re-introduce Fermi Problems.
- Guide students through a class Fermi problem.
- Discuss the solution to the class Fermi problem.

Day 1 Teacher Notes:

Part 1

- Write or project the Hot Cocoa part of the task (Part 1) for all to see. As a class, read aloud the task and supporting information. Provide groups with approximately one to two minutes of silent think time and an additional one to two minutes to share their thinking with their partner.
- Be careful to not give students too much of an idea of how to solve the hot cocoa problem. There are multiple approaches and entry points that students can take to correctly calculate a proportional ratio.
- During partner discussions, the teacher should circulate throughout the room and get a sense of the ideas students are generating. If partners have developed a solution, encourage them to come up with another solution. The multiple solutions will help students with their understanding of proportional relationships and creating equivalence.
- The teacher should allow different groups to share their ideas. This allows all groups the opportunity to recognize the different equivalent ratios and understand that there is more than one correct solution to the problem.
- Display the different solutions on the board using a variety of methods. Encourage students to think both in terms of proportional reasoning and pictorial reasoning.

Part 2

- Students will continue the task by starting Part 2. In this part, students are introduced to the type of ratio thinking useful for Fermi problems.
- Introduce students to Enrico Fermi, an Italian American physicist who developed the world’s first nuclear reactor. Enrico Fermi loved to think up and discuss problems that are impossible to directly measure but can be roughly estimated using known facts and calculations.
- Display a Fermi question that your students may find interesting and thought-provoking. Examples include…
    - If all students at your school stood fingertip to fingertip, how far would they stretch?
    - How many times does your heart beat in a year?
    - How many hours do you spend on your phone during your middle school years?
- A focus of this part of the task is to see the different ways to break a Fermi problem down into smaller questions that can be measured, estimated, or calculated. Students should use a copy of the graphic organizer to record their thinking.
- A few examples of productive sub-questions include:
    - What information do we know?
    - What information can be measured?
    - What information cannot be measured but can be calculated?
    - What assumptions should we make?

- Work through the class Fermi problem.
    - The problem can be worked through as a class, individually, or with partners.
    - Students will begin by estimating an answer and providing estimations they definitely know will be too low and estimations they definitely know will be too high. This will help students think of the reasonableness of their answers while solving.
    - Encourage students to think about the sub-questions they will need to measure, estimate, or calculate in order to solve the larger problem.
        - Record the sub-questions in the graphic organizer provided and label each sub-question in the order they should be answered.
        - When students have labeled the sub-questions, encourage them to think about whether there are any sub-question gaps that are missing. If there are, write additional questions to fill the gaps.
- Ask students to share some of their smaller questions. Then, discuss how you might come up with answers to these smaller questions, which likely revolve around what information is known, can be measured, or can be computed. Also, discuss how our assumptions about the situation affect how we solve the problem.
- Using ratio reasoning and the answer to the sub-questions, calculate the solution to the class Fermi problem.

Day 2

- Review the steps used in solving the class Fermi problem.
- Hand out the suggested Fermi problems collected earlier, with one question selected and starred for each group.
- Groups work to solve their problem.
- Groups make class display posters.
- Gallery walk of the posters
- Analyze the task. (See the discussion below.)
- Exit Slip

Day 2 Teacher Notes:

- Students should use the graphic organizer provided to organize and record their thinking.
- Students should measure, use drawings and tables, make estimates, and calculate.
- The teacher should be available to ask questions, answer questions, encourage, and support.
- The poster should be a visual display of the problem, focusing on the mathematics that was used. It should include the title of the Fermi problem that the group explored and should include a table and/or a tape or double number line diagram. Encourage students to provide enough details so others can follow their thinking easily.
- For the gallery walk, each group will "visit" posters made by other groups and leave Post-it comments focusing on how the mathematicians (the makers of the poster) used ratio reasoning. Then each group will review the comments as an evaluation of their work.
- Students should <u>individually</u> write their answers to this Exit Slip question on a Post-It Note and place it in a designated area: "How did your group use your knowledge of ratios and proportional reasoning to successfully calculate the answer to your Fermi problem?" The teacher can quickly read through them and share the top three to five responses with the class.

Share, Discuss, and Analyze Phase

Essential Understanding #1: *An understanding of ratios and proportional relationships, with an ability to apply that knowledge, is valuable in solving more complicated problems.*

Share- The lesson opens with the presentation of a Hot Cocoa problem, then moves to Fermi problems. Both types of problems provide students with opportunities to use ratio reasoning. The cocoa problem is a simpler problem. Students need to see that they use the same kind of reasoning as they move to the more complicated one.
Discuss- On the cocoa problem, student pairs need to discuss the ratio of cocoa to milk that is needed for good-tasting cocoa. With each new "recipe" that they find, they should refer back to the original ratio and discuss whether or not it is in the same ratio. They should be able to explain why their solution works and be able to show their solution with a diagram. When they move to the Fermi problem, they should be able to find an original ratio. (Arm span : person; # of notebooks : student; gallons of water : pool, etc.) Then they should discuss that they are using the same type of reasoning they used in the cocoa problem to scale this up.
Analyze - As a class processes the Hot Cocoa problem, they should discuss and diagram several solutions. They should affirm that there are multiple (an infinite number) solutions. Students should be able to justify their solutions. They should discuss different representations that can be used to do this. As they discuss the Fermi problem, they should be able to respond to the following questions: "How did you use ratio reasoning to explore your Fermi problem?" and "How did your group use your knowledge of ratios and ratio reasoning to successfully calculate the answer to your Fermi problem?"

Essential Understanding #2: *An ability to break complex problems down into simpler parts is important in solving a real-world problem about ratios and rates.*
Share- The students will first solve the Hot Cocoa problem, a task that requires them to use ratio reasoning. Then they will move to a Fermi problem that is much more complicated, but uses the same type of reasoning multiple times after the problem is broken into simpler parts.
Discuss- Students can be guided by the graphic organizer. First, they should state their problem. "What do we know?" "What are we trying to find out?" Then they should ask sub-questions. What steps do we need to take to get there?
Analyze – The class will first process the class Fermi problem. This shared experience will be helpful as they move to the group problem. An important question to discuss is, "How did we break the problem down to make it more manageable?" The analysis of the group problem will be more general in nature. Questions that can be asked are, "How did you break the problem down to make it more manageable?" "What were some situations where you had to make the problem simpler in order to proceed?" What were some instances where you had to make an estimate in order to proceed?" and "What are some instances when you had to figure out what additional information you needed? "

Task in Action

The video clips below provide a demonstration of the task being implemented in a classroom as it aligns to the Effective Mathematics Teaching Practice indicated. These clips should be used by the teacher to model the implementation of the task in his or her own classroom.

- Establish Mathematics Goals to Focus Learning
  - Video Clip
- Implement Tasks that Promote Reasoning and Problem Solving
  - Video Clip
- Pose Purposeful Questions
  - Video Clip 1
  - Video Clip 2
- Support Productive Struggle in Learning Mathematics
  - Video Clip
- Elicit Evidence of Student Thinking
  - Video Clip