Overview of the West Virginia College- and Career-Readiness Standards for Mathematics

Included in Policy 2520.2B, the West Virginia College- and Career-Readiness Standards for Mathematics are two types of standards: the Mathematical Habits of Mind and the grade-level or course-specific Mathematics Content Standards. These standards address the skills, knowledge, and dispositions that students should develop to foster mathematical understanding and expertise, as well as concepts, skills, and knowledge – what students need to understand, know, and be able to do. The standards also require that the Mathematical Habits of Mind and the grade-level or course-specific Mathematics Content Standards be connected. These connections are essential to support the development of students' broader mathematical understanding, as students who lack understanding of a topic may rely too heavily on procedures. The Mathematical Habits of Mind must be taught as carefully and practiced as intentionally as the grade-level or course-specific Mathematics Content Standards. Neither type should be isolated from the other; mathematics instruction is most effective when these two aspects of the West Virginia College- and Career-Readiness Standards for Mathematics come together as a powerful whole.

Mathematical Habits of Mind

Overarching Habits of Mind of a Productive Mathematical Thinker

MHM1
Make senses of problems and persevere in solving them

MHM6
Attend to precision

Reasoning and Explaining

MHM2
Reason abstractly and quantitatively

MHM3
Construct viable arguments and critique the reasoning of others

Modeling and Using Tools

MHM4
Model with mathematics

MHM5
Use appropriate tools strategically

Seeing Structure and Generalizing

MHM7
Look for and make use of structure

MHM8
Look for and express regularity in repeated reasoning

The eight Mathematical Habits of Mind (MHM) describe the attributes of mathematically proficient students and the expertise that mathematics educators at all levels should seek to develop in their students. The Mathematical Habits of Mind provide a vehicle through which students engage with and learn mathematics. As students move from elementary school through high school, the Mathematical Habits of Mind are integrated in the tasks as students engage in doing mathematics and master new and more advanced mathematical ideas and understandings.

The Mathematical Habits of Mind rest on important "processes and proficiencies " with longstanding importance in mathematics education. The first of these are the National Council of Teachers of Mathematics' process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report Adding it Up: adaptive reasoning, strategic competence, conceptual understanding, procedural fluency, and productive disposition (NGA/CCSSO 2010).

Ideally, several Mathematical Habits of Mind will be embedded in each lesson as they interact and overlap with each other. The Mathematical Habits of Mind are not a checklist; they are the basis for mathematics instruction and learning. To help students persevere in solving problems (MHM1), teachers need to allow their students to struggle productively, and they must be attentive to the type of feedback they provide to students. Dr. Carol Dweck's research (Dweck 2006) revealed that feedback offering praise of effort and perseverance seems to engender a "growth mindset." In Dweck's estimation, growth-minded teachers tell students the truth about being able to close the learning gap between them and their peers and then give them the tools to close the gap (Dweck 2006).

Students who are proficient in the eight Mathematical Habits of Mind are able to use these skills not only in mathematics, but across disciplines and into their lives beyond school, college, and career.

Policy 2520.2B

West Virginia College- and Career-Readiness Standards for Mathematics

Mathematical Habits of Mind

The Mathematical Habits of Mind (hereinafter MHM) describe varieties of expertise that mathematics educators at all levels should develop in their students.

MHM1. Make sense of problems and persevere in solving them.
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables and graphs or draw diagrams of important features and relationships, graph data and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MHM2. Reason abstractly and quantitatively.
Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize - to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand, considering the units involved, attending to the meaning of quantities, not just how to compute them, and knowing and flexibly using different properties of operations and objects.

MHM3. Construct viable arguments and critique the reasoning of others.
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a

flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense and ask useful questions to clarify or improve the arguments.

MHM4. Model with mathematics.
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

MHM5. Use appropriate tools strategically.
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

MHM6. Attend to precision.
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

MHM7. Look for and make use of structure.
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

MHM8. Look for and express regularity in repeated reasoning.
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$ and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Calculus Content Standards and Objectives

West Virginia teachers who provide mathematics instruction must integrate content standards with the MHM. Students will deepen and extend their understanding of functions, continuity, limits, differentiation, applications of derivatives, integrals, and applications of integration. Students will apply the Rule of Four (Numerical, Analytical, Graphical and Verbal) throughout the course and use available technology to enhance learning. Student will use graphing utilities to investigate concepts and to evaluate derivatives and integrals. The MHM, which should be integrated in these content areas, include: making sense of problems and persevering in solving them; reasoning abstractly and quantitatively; constructing viable arguments and critiquing the reasoning of others; modeling with mathematics; using appropriate tools strategically; attending to precision; looking for and making use of structure; and looking for and expressing regularity in repeated reasoning. Students will continue developing mathematical proficiency in a developmentally-appropriate progression of standards. Continuing the skill progressions from previous courses, the following chart represents the mathematical understandings that will be developed:

Algebra	Geometry
• A utility company burns coal to generate electricity. The cost C in dollars of removing p% of the air pollutants emissions is $C=\frac{90,000p}{100-p}$, $0\le p<100$. Find the cost of removing (a) 10%, (b) 25%, and (c) 75% of the pollutants. Find the limit of C as $p\to 100^{-}$. • A management company is planning to build a new apartment complex. Knowing the maximum number of apartments that the lot can hold and given a function for the maintenance costs, determine the number of apartments that will minimize the maintenance costs. • The velocity v of the flow of blood at a distance r from the central axis of an artery of radius R is $v=k\left(R^2-r^2\right)$ where k is the constant of proportionality. Find the average rate of flow of blood along a radius of the artery (use 0 and R as the limits of integration).	• The radius of a right circular cylindrical balloon is given by $\sqrt{t+2}$ and its height is $\frac{1}{2}\sqrt{t}$, where t is time in seconds and the dimensions are in inches. Find the rate of change of the volume with respect to time. • Given 50 meters of framing material, construct a window that will let in the most light if the middle of the window is a rectangle and the top and bottom of the window are semi-circles. • The graph of f consists of the three line segments joining the points $(0,0)$, $(2-2)$, $(6,2)$, and $(8,3)$. The function F is defined as follows $F(x)=\int_0^x f(t)\,dt$ Find the total enclosed areas generated by f and the x-axis. Determine the points of inflection of F on the interval $(0,8)$.

Data Analysis and Probability	
• The average data entry speeds S (words per minute) of a business student after t weeks of lessons are recorded in the following table. t: 5, 10, 15, 20, 25, 30 S: 28, 56, 79, 90, 93, 94 A model for the data is $S = \frac{100t^2}{65+t}, \quad t > 0.$ Do you think that there is a limiting speed? If so, what is the limiting speed? If not, why? • Identify a real-world situation that involves quantities that change over time and develop a method to collect and analyze related data. Develop a continuous function to model the data and generalize the results to make a conclusion. • A sheet of typing paper is ruled with parallel lines that are 2 inches apart. A two-inch needle is tossed randomly onto the sheet of paper. The probability that the needle will touch a line is $P = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta$ where θ is the acute angle between the needle and any one of the parallel lines. Find the probability.	

t	5	10	15	20	25	30
S	28	56	79	90	93	94

Numbering of Standards

The following Mathematics Standards will be numbered continuously. The following ranges relate to the clusters found within Calculus:

Algebra	
Understand the key concepts, connections, and applications of functions, limits, continuity, derivatives, and integrals represented in multiple ways.	Standards 1-19
Geometry	
Apply the key concepts, connections, and applications of limits, continuity, derivatives, and integration for a wide variety of regions.	Standards 20-22
Data Analysis and Probability	
Apply the key concepts and applications of limits, continuity, derivatives, and integration to analyze functions that represent a collection of data.	Standard 23

Algebra

Cluster	**Understand the key concepts, connections and applications of functions, limits, continuity, derivatives, and integrals represented in multiple ways.**
M.C.1	Use abstract notation to apply properties of algebraic, trigonometric, exponential, logarithmic, and composite functions, as well as their inverses, represented graphically, numerically, analytically, and verbally; and demonstrate an understanding of the connections among these representations.
M.C.2	Demonstrate a conceptual understanding of the definition of a limit via the analysis of continuous and discontinuous functions represented using multiple representations (e.g., graphs and tables).
M.C.3	Use the properties of limits including addition, product, quotient, composition, and squeeze/sandwich theorem to calculate the various forms of limits: one-sided limits, limits at infinity, infinite limits, limits that do not exist, and special limits such as $\lim_{x\to 0}\frac{\sin x}{x}=1$ and $\lim_{x\to 0}\frac{1-\cos x}{x}=0$.
M.C.4	Apply the definition of continuity to determine where a function is continuous or discontinuous including continuity at a point, continuity over an interval, application of the Intermediate Value Theorem, and graphical interpretation of continuity and discontinuity.
M.C.5	Investigate and apply the definition of the derivative graphically, numerically, and analytically at a point; conceptually interpret the derivative as an instantaneous rate of change and the slope of the tangent line.
M.C.6	Discriminate between the average rate of change and the instantaneous rate of change using real-world problems.
M.C.7	Recognize when the Extreme Value Theorem indicates that function extrema exist.

Cluster	**Understand the key concepts, connections and applications of functions, limits, continuity, derivatives, and integrals represented in multiple ways.**
M.C.8	Quickly recall and apply rules of differentiation including the constant multiple rule, the sum rule, the difference rule, the product rule, the quotient rule, the power rule, and the chain rule as applied to algebraic, trigonometric, exponential, logarithmic, and inverse trigonometric functions using techniques of both explicit and implicit differentiation.
M.C.9	Apply Rolle's Theorem and the Mean Value Theorem to real-world problems.
M.C.10	Construct and use mathematical models to solve optimization, related-rates, velocity, and acceleration problems.
M.C.11	Determine antiderivatives that follow from derivatives of basic functions and apply substitution of variables.
M.C.12	Evaluate definite integrals using basic integration properties such as addition, subtraction, constant multipliers, the power rule, substitution, and change of limits.
M.C.13	Characterize the definite integral as the total change of a function over an interval and use this to solve real-world problems.
M.C.14	Apply the Fundamental Theorem of Calculus to evaluate definite integrals and to formulate a cumulative area function and interpret the function as it relates to the integrand.
M.C.15	Use limits to deduce asymptotic behavior of the graph of a function.
M.C.16	Compare and contrast the limit definition (not delta epsilon) of continuity and the graphical interpretation of the continuity of a function at a point; recognize different types of discontinuities.
M.C.17	Develop tangent lines as best linear approximations to functions near specific points; explain this conceptually; construct these tangent lines; and apply this concept to Newton's Method.
M.C.18	Investigate and explain the relationships among the graphs of a function, its derivative and its second derivative; construct the graph of a function using the first and second derivatives including extrema, points of inflection, and asymptotic behavior.
M.C.19	Approximate areas under a curve using Riemann sums by applying and comparing left, right, and midpoint methods for a finite number of subintervals.

Geometry

Cluster	Apply the key concepts, connections and applications of limits, continuity, derivatives, and integration for a wide variety of regions.
M.C.20	Justify why differentiability implies continuity and classify functional cases when continuity does not imply differentiability.
M.C.21	Calculate a definite integral using Riemann sums by evaluating an infinite limit of a sum using summation notation and rules for summation.
M.C.22	Use integration to solve problems that involve linear displacement, total distance, position, velocity, acceleration, and area between curves by looking at both functions of x and functions of y; utilize units to interpret the physical nature of the calculus process.

Data Analysis and Probability

Cluster	Apply the key concepts and applications of limits, continuity, derivatives, and integration to analyze functions that represent a collection of data.
M.C.23	Identify a real-world situation that involves quantities that change over time; pose a question; make a hypothesis as to the answer; develop, justify, and implement a method to collect, organize, and analyze related data; extend the nature of collected, discrete data to that of a continuous function that describes the known data set; generalize the results to make a conclusion; compare the hypothesis and the conclusion; present the project numerically, analytically, graphically, and verbally using the predictive and analytic tools of calculus.